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ON A PROBLEM OF L. MIŠÍK

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Let X be a topological space, $\mathscr{B}$ a basis for the topology, and f a real valued function on X. In [2], L. Mišík defined the following notions:

$A \subset X$ has the property $M_1'(\mathscr{B})$ if, for each $B \in \mathscr{B}$ with $\bar{B} \cap A \neq \emptyset$, $B \cap A$ is uncountable, where $\bar{B}$ denotes the closure of B;

$f \in \mathfrak{M}_1'(\mathscr{B})$ if, for each real a, the sets $\{x : f(x) < a\}$ and $\{x : f(x) > a\}$ have the property $M_1'(\mathscr{B})$;

$f \in \mathscr{D}_0(\mathscr{B})$ if, for each $B \in \mathscr{B}$, $x, y \in \bar{B}$, real numbers α such that $f(x) < \alpha < f(y)$ and $\varepsilon > 0$, there exists $\xi \in B$ with $f(\xi) \in (\alpha - \varepsilon, \alpha + \varepsilon)$;

$f \in \mathscr{D}(\mathscr{B})$ if, for each $B \in \mathscr{B}$, $x, y \in \bar{B}$, real number α such that $f(x) < \alpha < f(y)$, there exists $\xi \in B$ with $f(\xi) = \alpha$. He proved that the classes $\mathfrak{M}_1'(\mathscr{B})$ and $\mathscr{D}_0(\mathscr{B})$ are closed under uniform convergence. Clearly $\mathscr{D}(\mathscr{B}) \subset \mathscr{D}_0(\mathscr{B})$. Also, $\mathscr{D}(\mathscr{B}) \subset \mathfrak{M}_1'(\mathscr{B})$ if each $B \in \mathscr{B}$ is uncountable. Thus, if X is a topological space in which open sets are uncountable, and if $\{f_n\}$ is a sequence in $\mathscr{D}(\mathscr{B})$ converging uniformly to f, then $f \in \mathfrak{M}_1'(\mathscr{B}) \cap \mathscr{D}_0(\mathscr{B})$. He raised the question whether every $f \in \mathfrak{M}_1'(\mathscr{B}) \cap \mathscr{D}_0(\mathscr{B})$ is the limit of a uniformly convergent sequence in $\mathscr{D}(\mathscr{B})$.

In the present paper, a negative answer to the above question is given. On the other hand, a subclass of $\mathfrak{M}_1'(\mathscr{B}) \cap \mathscr{D}_0(\mathscr{B})$ is found to be the uniform closure of $\mathscr{D}(\mathscr{B})$ under certain conditions in Theorem which includes a result in [1]. Furthermore, we discuss the possibility of a generalization to transformations on X.

In the sequel, we assume that each $B \in \mathscr{B}$ is uncountable.

Definition 1. *$\mathscr{V}(\mathscr{B})$ is the class of functions f such that, for each pair of numbers $a < b$, the set $\{x : a < f(x) < b\}$ has the property $M_1'(\mathscr{B})$.*

As Mišík did for the class $\mathfrak{M}_1'(\mathscr{B})$, we can easily show that $\mathscr{D}(\mathscr{B}) \subset \mathscr{V}(\mathscr{B})$ and $\mathscr{V}(\mathscr{B})$ is closed under uniform convergence. Consequently, if f is the limit of a uniformly convergent sequence in $\mathscr{D}(\mathscr{B})$, then $f \in \mathscr{V}(\mathscr{B}) \cap \mathscr{D}_0(\mathscr{B})$. Now we give a negative answer to the above mentioned question by constructing a function $f \in \mathfrak{M}_1'(\mathscr{B}) \cap \mathscr{D}_0(\mathscr{B}) - \mathscr{V}(\mathscr{B})$.

Let X be the real line, $\mathscr{B}$ the collection of all open intervals, A the set of all irrationals in $(0, 1)$ and H a countable dense subset of A. Enumerate all open intervals in $(0, 1)$ with rational endpoints as $\{J_n\}_{n=1}^{\infty}$. We can pick $x_1 \neq y_1$ in $H \cap J_1$ and $x_n \neq y_n$ in $H \cap J_n - \{x_i\}_{i=1}^{n-1} - \{y_i\}_{i=1}^{n-1}$ for $n > 1$. Let $H_1' = \{x_n : n = 1, 2, \ldots\}$ and $H_2' = \{y_n : n = 1, 2, \ldots\}$. Then $H_1' \cap H_2' = \emptyset$ and they are dense subsets of H. Applying the same process to H_2', we obtain disjoint dense subsets H_2 and H_3' of H_2'. By induction, for each $n > 1$, H_n' has two disjoint dense subsets H_n and H_{n+1}'. Let $H_1 = H - \bigcup_{n=2}^{\infty} H_n$ (it should be noted that $H_1 \supset H_1'$). Then $H = \bigcup_{n=1}^{\infty} H_n$ and $\{H_n\}$ is a sequence of mutually disjoint dense subsets of $(0, 1)$. By Lemma 4.1 in [1], $A - H = A_1 \cup A_2$, where $A_1 \cap A_2 = \emptyset$ and A_1, A_2 are c-dense in $A - H$, that is, for $i = 1, 2$, the set $A_i \cap U$ is uncountable whenever U is an open set with $U \cap \cap (A - H) \neq \emptyset$. We define f as follows:

$$
\begin{aligned}
f(x) &= x \quad \text{if} \quad x \in X - A\,, \\
&= 0 \quad \text{if} \quad x \in A_1\,, \\
&= 1 \quad \text{if} \quad x \in A_2\,, \\
&= r_n \quad \text{if} \quad x \in H_n\,,
\end{aligned}
$$

where $\{r_n\}$ is an enumeration of all rationals in $(0, 1)$. It can be checked without difficulty that $f \in \mathfrak{M}_1'(\mathscr{B}) \cap \mathscr{D}_0(\mathscr{B})$. However, $\{x : 0 < f(x) < 1\} = H \cup ((0, 1) - A)$ does not have the property $M_1'(\mathscr{B})$. Therefore $f \notin \mathscr{V}(\mathscr{B})$.

Definition 2. $\mathscr{U}(\mathscr{B}) = \mathscr{V}(\mathscr{B}) \cap \mathscr{D}_0(\mathscr{B})$.

Let $\operatorname{Card} X$ denote the cardinality of X, and c that of the continuum. Suppose $\operatorname{Card} X = c$ and $\operatorname{Card} \mathscr{B} \leq c$. Then we have the following

Theorem. $f \in \mathscr{U}(\mathscr{B})$ *if and only if it is the limit of a uniformly convergent sequence in* $\mathscr{D}(\mathscr{B})$.

To prove the theorem, we need a lemma which follows immediately from Lemma 1 in [3]:

Lemma. *Let A be a set with* $\operatorname{Card} A = c$, *$\mathscr{F}$ a family of subsets of A such that* $0 < \operatorname{Card} \mathscr{F} \leq c$ *and* $\operatorname{Card} F = c$ *for each* $F \in \mathscr{F}$. *Then there exist pairwise disjoint sets* $A^0, A^1, \ldots, A^\mu, \ldots$ ($\mu < \Omega$, *the first ordinal number corresponding to* c) *in A such that*

1° $\bigcup \{A^\mu : 0 \leq \mu < \Omega\} = A$,

2° $\operatorname{Card} F \cap A^\mu = c$ for every $F \in \mathscr{F}$ and every $\mu < \Omega$.

Proof of Theorem. The "if" part was already mentioned preceding the example. Now we assume that $f \in \mathscr{U}(\mathscr{B})$ and $\varepsilon > 0$. It is sufficient to show that there exists $g \in \mathscr{D}(\mathscr{B})$ such that $|f(x) - g(x)| < \varepsilon$ for every $x \in X$.

The real line can be decomposed as $\bigcup_{n=1}^{\infty} I_n$, where each $I_n = [k\varepsilon, (k+1)\varepsilon)$ for some integer k. Let $A_n = f^{-1}(I_n^0)$, where I_n^0 is the interior of I_n. Since $f \in \mathscr{U}(\mathscr{B}) \subset \mathscr{V}(\mathscr{B})$, for each $B \in \mathscr{B}$ and each n, we have either $\bar{B} \cap A_n = \emptyset$ or Card $B \cap A_n = c$. For the n's such that $A_n \neq \emptyset$, we have Card $A_n = c$, $0 <$ Card $\mathscr{F}_n \leqq c$, where $\mathscr{F}_n = \{B \cap A_n : B \in \mathscr{B}$ and $\bar{B} \cap A_n \neq \emptyset\}$, and Card $B \cap A_n = c$ for each $B \cap A_n \in \mathscr{F}_n$. By the lemma, there are pairwise disjoint sets A_n^μ, $0 \leqq \mu < \Omega$, such that $A_n = \bigcup\{A_n^\mu : 0 \leqq \mu < \Omega\}$ and Card $B \cap A_n^\mu = c$ for each $B \in \mathscr{B}$ with $\bar{B} \cap A_n \neq \emptyset$ and each μ, $0 \leqq \mu < \Omega$.

For each n with $A_n \neq \emptyset$, let T_n be an onto map from $\{\mu : 0 \leqq \mu < \Omega\}$ to $\bar{I}_n$ and let g_n be defined on A_n by

$$g_n(x) = T_n(\mu) \quad \text{if} \quad x \in A_n^\mu .$$

Clearly $g_n(B \cap A_n) = \bar{I}_n$ for each $B \in \mathscr{B}$ such that $\bar{B} \cap A_n \neq \emptyset$. We define g as follows:

$$\begin{aligned} g(x) &= g_n(x) \quad \text{if} \quad x \in A_n \quad \text{for some } n, \\ &= f(x) \quad \text{otherwise}. \end{aligned}$$

It is immediate that $|f(x) - g(x)| < \varepsilon$ for every $x \in X$. Now we prove that $g \in \mathscr{D}(\mathscr{B})$. Let $B \in \mathscr{B}$, $x, y \in \bar{B}$ and $g(x) < \alpha < g(y)$ be given. We want to show that $\alpha \in g(B)$. If $f(x) < \alpha < f(y)$, $\alpha \in I_n$ for some n, then there exists $r \in I_n^0$ such that $f(x) < r < f(y)$. It follows from $f \in \mathscr{U}(\mathscr{B}) \subset \mathscr{D}_0(\mathscr{B})$ that $B \cap A_n \neq \emptyset$. Thus $\alpha \in I_n \subset \bar{I}_n = g_n(B \cap A_n) \subset g(B)$. If $\alpha \leqq f(x)$, then $g(x) < \alpha \leqq f(x)$ and hence there must be some n_1 such that $x \in A_{n_1}$. Now we have $g(x) \in \bar{I}_{n_1}$, $f(x) \in I_{n_1}^0$, and $x \in \bar{B} \cap A_{n_1}$. Consequently, $\alpha \in \bar{I}_{n_1} = g_{n_1}(B \cap A_{n_1}) \subset g(B)$. If $\alpha \geqq f(y)$, then $f(y) \leqq \alpha < g(y)$ and $y \in A_{n_2}$ for some n_2. Similar to the above, $\alpha \in \bar{I}_{n_2} \subset g(B)$. The proof is completed.

Remark 1. From Definition 2, we can easily prove that $f \in \mathscr{U}(\mathscr{B})$ if and only if, for each $B \in \mathscr{B}$, $x, y \in \bar{B}$ and each countable set $D \subset B$, $J \cap f(B - D)$ is dense in J, where J is the interval with $f(x)$ and $f(y)$ as endpoints. If X and $\mathscr{B}$ are taken as the real line and the collection of all open intervals respectively, then $\mathscr{U}(\mathscr{B})$ becomes the class $\mathscr{U}$ in [1]. Thus the above theorem includes the corresponding result in [1].

Remark 2. The definitions of the classes $\mathscr{D}_0(\mathscr{B})$, $\mathscr{D}(\mathscr{B})$, $\mathscr{V}(\mathscr{B})$ can be given in the following version by which we can generalize these concepts to transformations from X to a topological space Y:

$f \in \mathscr{D}_0(\mathscr{B})$ if, for each $B \in \mathscr{B}$, $\overline{f(\tilde{B})}$ is connected whenever $\tilde{B}$ is a set such that $B \subset \tilde{B} \subset \bar{B}$;

$f \in \mathscr{D}(\mathscr{B})$ if, for each $B \in \mathscr{B}$, $f(\tilde{B})$ is connected whenever $\tilde{B}$ is a set such that $B \subset \tilde{B} \subset \bar{B}$;

$f \in \mathscr{V}(\mathscr{B})$ if, for each $B \in \mathscr{B}$ and each open set V in Y such that $\bar{B} \cap f^{-1}(V) \neq \emptyset$, Card $B \cap f^{-1}(V) = c$.

Let Y be a fixed metric space. Consider the classes of transformations from X to Y. It can be shown that $\mathscr{D}(\mathscr{B}) \subset \mathscr{U}(\mathscr{B})$ which is defined to be $\mathscr{V}(\mathscr{B}) \cap \mathscr{D}_0(\mathscr{B})$, and $\mathscr{V}(\mathscr{B})$ is closed under uniform convergence. It is interesting to know whether $\mathscr{D}_0(\mathscr{B})$ is still closed under uniform convergence and what is the uniform closure of the class $\mathscr{D}(\mathscr{B})$.

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