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Radii of starlikeness and coefficient estimates of a class of analytic functions

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1. INTRODUCTION

Let $S^*$ denote the class of functions $f(z)$ analytic in the open unit disc $E\{z : |z| < 1\}$, normalized so that $f(0) = 0 = f'(0) - 1$ and univalently starlike in $E$. The properties of the elements of this class have been investigated extensively for many years. One of the more important early discoveries for the class $S^*$ was that $f(z)$ satisfies the inequality

$$|\sqrt{(z/f(z)) - 1}| > 1, \quad (z \in E).$$

This fact may also be expressed in the form

$$\text{Re} \sqrt{(f(z)/z)} \geq \frac{1}{1 + |z|} > 1/2, \quad (z \in E).$$

Then $f(z)/z \ll (1 + z)^{-2}$ in $E$ (where $\ll$ denotes subordination) and there exists an analytic function $\omega(z)$, $|\omega(z)| \leq |z| < 1$, such that

$$f(z)/z = \frac{1}{(1 + \omega(z))^2}, \quad (z \in E).$$

Proofs of this attractive result are due to Marx [4], Strohhächer [8], and to Robertson [6]. Motivated by this discovery, we introduce the class $S(\alpha, \beta)$ as follows.

A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disc $E$ is in the class $S(\alpha, \beta)$ if it satisfies the condition

$$f(z)/z \ll \left[\frac{1 + (2\alpha \beta - 1)z}{1 + (2\beta - 1)z}\right]^2, \quad (z \in E),$$

where $\alpha$ and $\beta$ are arbitrary fixed numbers, $0 \leq \alpha < 1, 0 < \beta \leq 1.$
It follows from the definition of subordination that \( f \in S(\alpha, \beta) \) has a representation of the form
\[
\frac{f(z)}{z} = \left[ \frac{1 + (2\alpha\beta - 1) \omega(z)}{1 + (2\beta - 1) \omega(z)} \right]^2, \quad (z \in E)
\]
for some function \( \omega \), analytic in \( E \), and satisfying the conditions \( \omega(0) = 0 \) and \( |\omega(z)| < 1, \; z \in E \).

A function \( f \in S(\alpha, \beta) \) may not be univalently starlike in \( E \) as is easily seen from the example \( f(z) = z(1 + z^2)^{-2} \in S(1/2, 1) \).

The class \( S(1/2, 1) \) has been investigated by Dvořák [2], Duren and Schober [1], and Reade and Umezawa [5]. We, further, note that the class \( S(1/(2\varrho), 1) \equiv S(1/(2\varrho)), \; \varrho > 1/2, \) is larger than the class introduced and studied by Goel [3].

In this paper, we obtain the radii of starlikeness and coefficient estimates for the functions in the class \( S(\alpha, \beta) \).

2. RADII OF STARLIKENESS

Let \( B \) denote the class of analytic functions \( \omega \) in \( E \) which satisfy the conditions (i) \( \omega(0) = 0 \), and (ii) \( |\omega(z)| < 1 \) for \( z \) in \( E \).

**Theorem 1.** Let \( f \in S(\alpha, \beta) \) and let \( r_0 \) be the smallest positive root of the equation
\[
(2.1) \quad (2\beta - 1) (2\alpha\beta - 1) r^4 - 2(2\beta - 1) (2\alpha\beta - 1) r^3 - 2(\beta + \alpha\beta + 2\alpha\beta^2 - 1) r^2 - 2r + 1 = 0 .
\]
Then

(i) for \( 0 \leq r < r_0 \), \( f \) is starlike in \( |z| < r_1 \), where \( r_1 \) is the smallest positive root of the equation
\[
(2.2) \quad (2\beta - 1) (2\alpha\beta - 1) r^2 + 2(3\alpha\beta - \beta - 1) r + 1 = 0 ,
\]

(ii) for \( r_0 \leq r < 1 \), \( f \) is starlike in \( |z| < r_2 \), where \( r_2 \) is the smallest positive root of the equation
\[
(2.3) \quad (16\alpha\beta - 9 - \alpha) r^4 - 2(8\alpha\beta + 3 - 3\alpha) r^2 + (9\alpha - 1) = 0 .
\]
The bounds for \( |z| \) in (i) and (ii) are sharp.

**Proof.** If \( f(z) = z + a_2 z^2 + \ldots \) and
\[
(2.4) \quad p(z) = \left( \frac{f(z)}{z} \right)^{1/2},
\]
then \( p(z) \) is analytic in \( E \) and \( p(0) = 1 \). Thus (1.3) may be rewritten as
\[
(2.5) \quad p(z) = \frac{1 + (2\alpha\beta - 1) \omega(z)}{1 + (2\beta - 1) \omega(z)} ,
\]
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where \( \omega \in B \). Taking logarithmic derivatives of (2.5), we find that

\[
\text{Re} \left\{ \frac{z^2}{p(z)} \right\} = -2\beta(1 - \alpha) \text{Re} \left\{ \frac{z \omega'(z)}{(1 + (2\beta - 1) \omega(z))(1 + (2\alpha\beta - 1) \omega(z))} \right\}.
\]

From (2.4) we may write

\[
\text{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} = 1 + 2 \text{Re} \left\{ \frac{z p'(z)}{p(z)} \right\}.
\]

Combining (2.6) and (2.7), we get

\[
\text{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} = 1 - 4 \beta(1 - \alpha) \frac{z \omega'(z)}{(1 + (2\beta - 1) \omega(z))(1 + (2\alpha\beta - 1) \omega(z))}.
\]

It is well known [7] that if \( \omega \in B \), then for all \( z \in E \),

\[
|z \omega'(z) - \omega(z)| \leq \frac{|z|^2 - |\omega(z)|^2}{1 - |z|^2}.
\]

Equation (2.8) yields in conjunction with (2.9),

\[
\text{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} \geq 1 + \frac{1}{\beta(1 - \alpha)} \left[ \text{Re} \left\{ (2\beta - 1) p(z) + \frac{2\alpha\beta - 1}{p(z)} \right\} - \frac{2(\beta + \alpha\beta - 1)}{\beta(1 - \alpha)} r^2 \left( (2\beta - 1) p(z) - (2\alpha\beta - 1) \right)^2 - \frac{1 - |p(z)|^2}{1 - |z|^2} \right],
\]

where \( r = |z| \), \( z \in E \).

Noting that the transformation (2.5) maps the disc \( |\omega(z)| \leq r \) onto the disc \( |\omega(z) - a| < d \), where

\[
a = \frac{1 - (2\beta - 1)(2\alpha\beta - 1) r^2}{1 - (2\beta - 1)^2 r^2}, \quad d = \frac{2\beta(1 - \alpha) r}{1 - (2\beta - 1)^2 r^2},
\]

we set \( p(z) = a + u + iv \) and \( R = |p(z)| \) in (2.10). Taking \( M(u, v) \) as the expression on the right hand side of (2.10), we get

\[
M(u, v) = \frac{u}{\beta(1 - \alpha)} \left[ (2 - \beta - 3\alpha\beta) + (2\beta - 1)(a + u) + \frac{(2\alpha\beta - 1)(a + u)}{R^2} \right] - \frac{(1 - (2\beta - 1)^2 r^2)(d^2 - u^2 - v^2)}{1 - r^2}.
\]

By differentiating (2.11) partially with respect to \( v \), we obtain

\[
\frac{\partial M(u, v)}{\partial v} = \frac{v R^{-4} N(u, v)}{\beta(1 - \alpha)},
\]

where...
where
\[ N(u, v) = 2(1 - 2\alpha\beta) (a + u) + \frac{(1 - (2\beta - 1)^2 r^2)(d^2 - u^2 - v^2)}{1 - r^2} + \frac{2(1 - (2\beta - 1)^2 r^2) R^3}{1 - r^2}. \]

It is easily seen that \( N(u, v) > 0 \), and so the minimum of \( M(u, v) \) on every chord \( u = \text{constant} \) is attained on the diameter \( v = 0 \). Taking \( v = 0 \) in (2.11), we get
\[
L(R) = M(R, 0) = \frac{2 - \beta - 3\alpha\beta}{\beta(1 - \alpha)} + \frac{2}{\beta(1 - \alpha)(1 - r^2)} \cdot \{\beta(1 - (2\beta - 1) r^2) R + \alpha\beta(1 - (2\alpha\beta - 1) r^2) R^{-1} - a(1 - (2\beta - 1)^2 r^2)\},
\]
where \( a - d \leq R \leq a + d \). Now it is easy to see that the absolute minimum of \( L(R) \) in \((0, \infty)\) is attained at
\begin{equation}
R_0 = \left(\frac{\alpha(1 - (2\alpha\beta - 1) r^2)}{1 - (2\beta - 1) r^2}\right)^{1/2},
\end{equation}
and equals
\begin{equation}
L(R_0) = 1 + \frac{2\mu(r, \alpha, \beta)}{(1 - \alpha)(1 - r^2)},
\end{equation}
where
\[
\mu(r, \alpha, \beta) = 2(\alpha(1 - (2\beta - 1) r^2)(1 - (2\alpha\beta - 1) r^2))^{1/2} - (1 + \alpha) + (4\alpha\beta - \alpha - 1) r^2.
\]

We note that \( R_0 < a + d \). However, \( R_0 \) may not always be greater than \( a - d \). Hence, when \( R_0 \in (0, a - d] \), the minimum of \( L(R) \) is attained at
\begin{equation}
R_1 = a - d = \frac{1 + (2\alpha\beta - 1) r}{1 + (2\beta - 1) r},
\end{equation}
and is equal to
\begin{equation}
L(R_1) = 1 - \frac{4\beta(1 - \alpha) r}{(1 + (2\beta - 1) r)(1 + (2\alpha\beta - 1) r)}.
\end{equation}

The two minima given by (2.13) and (2.15) coincide for such values of \( \alpha, \beta \) \((0 \leq \alpha < 1, 0 < \beta \leq 1)\) for which \( R_0 = R_1 \), which implies (2.1). We thus conclude that
\begin{equation}
\text{Re} \left( \frac{z f'(z)}{f(z)} \right) \geq \begin{cases} 
1 + \frac{2\mu(r, \alpha, \beta)}{(1 - \alpha)(1 - r^2)}, & R_0 \geq R_1, \\
1 - \frac{4\beta(1 - \alpha) r}{(1 + (2\beta - 1) r)(1 + (2\alpha\beta - 1) r)}, & R_0 \leq R_1.
\end{cases}
\end{equation}
Therefore the function $f$ is starlike if

\begin{align*}
(2.17) \quad 2\mu(r, \alpha, \beta) + (1 - \alpha)(1 - r^2) > 0, \quad R_0 \geq R_1, \\
(2.18) \quad (1 + (2\beta - 1) r)(1 + (2\alpha\beta - 1) r) - 4\beta(1 - \alpha) r > 0, \quad R_0 \leq R_1.
\end{align*}

Now it is easy to see that (2.18) and (2.17) are satisfied, respectively, for $|z| < r_1$ and $|z| < r_2$, where $r_1$ and $r_2$ are the smallest positive roots of the equations (2.2) and (2.3). This completes the proof of the theorem.

The functions given by

\begin{align*}
f_1(z) &= z \left\{ \frac{1 + (2\alpha\beta - 1) z}{1 + (2\beta - 1) z} \right\}^2, \\
f_2(z) &= z \left\{ \frac{1 - 2\alpha\beta z + (2\alpha\beta - 1) z^2}{1 - 2\beta bz + (2\beta - 1) z^2} \right\},
\end{align*}

where $b$ is determined by the relation

\[ \frac{1 - 2\alpha\beta br + (2\alpha\beta - 1) r^2}{1 - 2\beta br + (2\beta - 1) r^2} = R_0 = \left\{ \frac{\alpha(1 - (2\alpha\beta - 1) r^2)}{1 - (2\beta - 1) r^2} \right\}^{1/2}, \]

show, respectively, that the bounds in $|z|$ for (i) and (ii) are sharp for all admissible values of $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$.

The following Corollary arises from Theorem 1 by an easy computation.

**Corollary.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, analytic in $E$, satisfy the inequality

\[ \text{Re} \sqrt{\frac{\text{Re}(f(z))}{z}} > \frac{1}{2q}, \quad (q > 1/2) \]

for all $z$ in $E$. Let $q_0 > 1/2$ denote the smallest positive root of the equation

\[ 32q^3 - 104q^2 + 98q - 27 = 0. \]

Then

(i) for $1/2 < q \leq q_0$, $f$ is starlike in

\[ |z| < \left\{ \frac{8\sqrt{(4q - 2) - (6q + 5)}}{18q - 17} \right\}^{1/2}, \]

(ii) for $q \geq q_0$, $f$ is starlike in

\[ |z| < \frac{\sqrt{(20q^2 - 28q + 9) - (4q - 3)}}{2(q - 1)} . \]
These bounds for $|z|$ are sharp for the functions given by

$$f_1(z) = z \left\{ \frac{1 + (1/q - 1)z}{1 + z} \right\}^2,$$

$$f_2(z) = z \left\{ \frac{1 - (1/q)bz + (1/q - 1)z^2}{1 - 2bz + z^2} \right\}^2,$$

where $b$ is determined by the equation

$$\frac{1 - (1/q)br + (1/q - 1)r^2}{1 - 2br + r^2} = \left\{ \frac{1 - (1/q - 1)r^2}{2q(1 - r^2)} \right\}^{1/2}.$$

Goel [3] has proved the above result for the case of $q \geq 1$.

3. COEFFICIENT ESTIMATES

Theorem 2. Let $f(z) = z + az^2 + \ldots$ be in $S(\alpha, \beta)$. Then

$$|a_n| \leq 4\beta(1 - \alpha) \left\{ 1 - 2\beta(1 - \alpha) + \beta(1 - \alpha) n \right\}, \quad (n \geq 2)$$

for all values of $\alpha, \beta$ ($0 \leq \alpha < 1$, $0 < \beta \leq 1$). The result is sharp.

Proof. Letting

$$p(z) = \frac{1 + (2\beta - 1)\omega(z)}{1 + (2\beta - 1)\omega(z)} = 1 + p_1z + \ldots,$$

we may rewrite (1.3) as

$$z + z_2z^2 + \ldots = z[1 + p_1z + \ldots]^2.$$

Equating the coefficients of $z^{2m}$ and $z^{2m+1}$, we get

$$a_{2m+1} = p_m^2 + 2p_{2m} + 2 \sum_{r+s=2m} p_r p_s,$$

and

$$a_{2m+2} = 2p_{2m+1} + 2 \sum_{r+s=2m+1} p_r p_s, \quad (m = 1, 2, \ldots).$$

Further, (3.2) gives

$$\sum_{k=1}^{\infty} (2\beta - 1) p_kz^k \omega(z) = -\sum_{k=1}^{\infty} p_kz^k.$$

We observe that the coefficient $p_n$ on the right of (3.5) depends only on $p_1, p_2, \ldots, p_{n-1}$ on the left of (3.5). Hence for $n \geq 1$, it follows that

$$\left\{ 2\beta(1 - \alpha) + \sum_{k=1}^{n-1} (2\beta - 1) p_kz^k \right\} \omega(z) = -\sum_{k=1}^{n} p_kz^k - \sum_{k=n+1}^{\infty} d_kz^k,$$
where $\sum_{k=n+1}^{\infty} d_k z^k$ converges in $E$. Then

\begin{equation}
(3.6) \quad |2\beta(1 - \alpha) + \sum_{k=1}^{n-1} (2\beta - 1) p_k z^k| \geq \left| \sum_{k=1}^{n} p_k z^k + \sum_{k=n+1}^{\infty} d_k z^k \right|.
\end{equation}

Squaring both sides of (3.6), integrating round $|z| = r$, $0 < r < 1$, and finally taking the limit as $r \to 1$, we get

\[ 4\beta^2(1 - \alpha)^2 + \sum_{k=1}^{n-1} (2\beta - 1)^2 |p_k|^2 \geq \left| \sum_{k=1}^{n} p_k^2 + \sum_{k=n+1}^{\infty} |p_k|^2 \right|. \]

Simplifying and using the relation $0 < \beta \leq 1$, we obtain

\begin{equation}
(3.7) \quad |p_n| \leq 2\beta(1 - \alpha), \quad (n \geq 1).
\end{equation}

Using (3.7) in (3.3) and (3.4), we obtain

\begin{equation}
(3.8) \quad |a_{2m+1}| \leq 4\beta(1 - \alpha) + 8\beta^2(1 - \alpha)^2 \left( \frac{2m + 1 - 2}{2} \right),
\end{equation}

\begin{equation}
(3.9) \quad |a_{2m+2}| \leq 4\beta(1 - \alpha) + 8\beta^2(1 - \alpha)^2 \left( \frac{2m + 2 - 2}{2} \right).
\end{equation}

Combining (3.8) and (3.9) we have

\[ |a_n| \leq 4\beta(1 - \alpha) + 8\beta^2(1 - \alpha)^2 \left( \frac{n - 2}{2} \right), \]

which yields (3.1).

The equality in (3.1) holds for the function given by

\[ f(z) = z \left[ \frac{1 - (2\alpha \beta - 1)z}{1 - (2\beta - 1)z} \right]^2. \]

Remark. Setting $\alpha = 1/2$ and $\beta = 1$ in Theorem 2, we get $|a_n| \leq n$, $(n \geq 2)$. This result was obtained by Dvořák [2]. Further, replacing $\alpha$ by $1/(2\alpha)$ and setting $\beta = 1$ in Theorem 2 we have a result obtained in [3].

References


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