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Časopis pro pěstování matematiky, Vol. 108 (1983), No. 3, 272--284

Persistent URL: `http://dml.cz/dmlcz/118175`

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EXISTENCE AND MULTIPLICITY RESULTS FOR SOME WEAKLY NONLINEAR ELLIPTIC PROBLEMS AT RESONANCE

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(Received November 2, 1982)

1. INTRODUCTION

Let $\Omega \subset R^N$ be a bounded domain with boundary $\partial\Omega$, $\mathscr{L}$ a uniformly elliptic differential operator, λ_k any eigenvalue of $\mathscr{L}u + \lambda u = 0$ with zero Dirichlet boundary conditions. In this paper we study the nonlinear Dirichlet problem

$$\begin{aligned} \mathscr{L}u + \lambda_k u + g(x, u) &= f(x) \quad \text{on} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.1}$$

Especially, we are interested in a bounded nonlinearity g which does not satisfy the Landesman-Lazer condition, i.e.

$$\lim_{s \to \pm\infty} g(x, s) = 0. \tag{1.2}$$

This case has been studied by many authors. We refer the reader to [2–4], [6–11]. The results obtained can be divided into three parts:

1. Existence results.
2. Multiplicity results.
3. Results either dealing with the first simple eigenvalue or dealing with an arbitrary eigenvalue.

The papers [6], [8], [9], [10] and [11] deal with existence results for the problem (1.1). The works [6] and [11] treat the case of the second order elliptic differential operator $\mathscr{L}$ and $\lambda_k = \lambda_1$, i.e. the first simple eigenvalue with the corresponding positive eigenfunction. The nonlinear function g is supposed to be independent of x and to satisfy

$$s\, g(s) \leqq 0 \quad (\text{or } s\, g(s) \geqq 0). \tag{1.3}$$

The authors of [9] studied the case of the differential elliptic operator $\mathscr{L}$ of order $2m$ $(m \geqq 1)$, the general eigenvalue λ_k and an odd nonlinearity g independent of x.

In order to prove existence results for (1.1) they used the condition

$$\lim_{s \to \pm\infty} s\, g(s) = +\infty . \tag{1.4}$$

This condition is weakened to

$$\lim_{s \to \pm\infty} s\, g(s) > 0 \tag{1.5}$$

in [10], while the unique continuation property of the linear part is assumed (for the precise meaning of this notion see Section 2). The paper [8] deals with the symmetric differential operator of an arbitrary order $2m$ ($m \geqq 1$), the general eigenvalue λ_k and an odd nonlinearity g independent of x. The function g satisfies neither (1.3) nor (1.4) but some bound on the derivative g' is necessary.

The papers [2–4] and [7] deal with existence and multiplicity results for (1.1). However, the authors of [2–4] used a condition similar to (1.5) in order to prove existence and multiplicity results for nonlinearities which satisfy (1.2). The condition (1.5) is removed in [4], [7] but there the case of the first eigenvalue with the corresponding positive eigenfunction is studied.

The purpose of this paper is to study the range of the weakly nonlinear operator on the left hand side of (1.1) with the corresponding Dirichlet boundary conditions. Existence, nonexistence and multiplicity results are given for bounded nonlinearities which do not satisfy the Landesman-Lazer condition and which need not satisfy a condition of the type (1.5) imposed on the speed of convergence of $g(x, s)$ to zero when s tends to plus or minus infinity. The price we must pay for this generalization is some a priori bound on the derivative g' (see Open problem 6.2). This paper can be understood as a completion of the preceding papers mentioned above.

The idea of the proofs is to use the global Lyapunov-Schmidt method and the Brouwer degree theory. We assume that the reader is somewhat familiar with the works of Ambrosetti and Mancini [3], [4] and we refer him to these papers.

The paper is organized as follows. Section 2 contains the assumptions, the description of the problem and a preliminary lemma, where the equation is studied in the cokernel of the linear part. The proof of this lemma is ommited and it can be found in [4]. In Section 3 the main existence result is stated. The existence of multiple solutions is investigated in Section 4, where also the case of the simple eigenvalue is studied separately. Section 5 contains some applications of the abstract results stated above and in the last Section 6 we formulate some open problems the solution of which would lead, in the author's opinion, to a better understanding of weakly nonlinear problems of the type (1.1) with nonlinearities without Landesman-Lazer condition.

2. ASSUMPTIONS, FORMULATION OF THE PROBLEM

Let Ω be a bounded domain in $\mathbb{R}^N$ with lipschitzian boundary $\partial\Omega$ and let $E = W_0^{m,2}(\Omega)$. In the same way as in [3], [4], we will denote $\|\cdot\|$ the norm in E, by $\|\cdot\|_0$ the norm in $L^2(\Omega)$ and by $(\cdot, \cdot)_m$, or $(\cdot, \cdot)$, the scalar product in E, or in $L^2(\Omega)$, respectively. Let us consider the formal differential operator

$$\mathscr{L} = - \sum_{|\alpha|=|\beta|=m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta} D^\beta) .$$

We assume that $a_{\alpha\beta} = a_{\beta\alpha} \in L^\infty(\Omega)$ and there exists such a constant $\gamma > 0$ that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \xi^\alpha \xi^\beta \geqq \gamma |\xi|^{2m}$$

for each $\xi \in \mathbb{R}^N$. For $u, v \in E$, set

$$((u, v)) = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^\alpha u D^\beta v \, dx .$$

Let us consider the linear operator $L : E \to E$ defined by $(Lu, v)_m = -((u, v))$. L is a selfadjoint operator with infinitely many eigenvalues $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots$ and a corresponding complete orthonormal system of eigenfunctions $\varphi_1, \varphi_2, \ldots$. It is known that each λ_k has the variational characterization

$$\lambda_k = \min \left\{ \frac{((v, v))}{\|v\|_0^2} ;\ v \in E,\ (v, \varphi_i) = 0,\ i = 1, 2, \ldots, k-1 \right\} .$$

Let us denote by L_k the linear operator defined by

$$(L_k u, v)_m = (Lu, v)_m + \lambda_k (u, v) .$$

Then L_k is a Fredholm mapping of index zero.

We suppose that λ_k is an eigenvalue of multiplicity $p \geqq 1$, i.e. $\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+p-1} < \lambda_{k+p}$. Analogously as in [3] we set $V = \operatorname{Ker} L_k$, $V^\perp$ its orthogonal complement in such a way that $E = V \oplus V^\perp$ and $u \in E$ can be written in the form $u = v + w$, where $v \in V$ and $w \in V^\perp$.

We assume that the following *unique continuation property* holds: for every $v \in V$, $v \neq 0$ the set $\{x \in \Omega;\ v(x) = 0\}$ has zero Lebesque measure.

It appears (see [12]) that this form of unique continuation property is reasonable.

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function which is measurable in $x \in \Omega$ for all $s \in \mathbb{R}$ and continuously differentiable in s for almost all $x \in \Omega$. Moreover we assume:

(g1) there exists $M > 0$ such that

$$|g(x, s)| \leqq M$$

for all $(x, s) \in \Omega \times \mathbb{R}$;

(g2) $\int_0^{-\infty} g(x, s) \, ds = \int_0^{+\infty} g(x, s) \, ds$ for almost all $x \in \Omega$ and the integral $\int_\Omega dx \,.\, \int_0^{+\infty} g(x, s) \, ds \in \mathbb{R} \cup \{\pm\infty\}$ exists (we take Lebesgue integrals);

(g3) $\lambda_{k-1} < \text{const} \leqq \lambda_k + g'_s(x, s) \leqq \text{const} < \lambda_{k+p}$ if $k > 1$, $\text{const} \leqq g'_s(x, s) + \lambda_1 \leqq \text{const} < \lambda_{p+1}$ if $k = 1$,

for all $(x, s) \in \Omega \times \mathbb{R}$.

If g satisfies (g1), we can define a mapping $G : E \to E$ by

$$(Gu, v)_m = (g(x, u), v) \quad \text{for all} \quad v \in E\,,$$

and G is C^1, i.e. G is continuously Fréchet differentiable.

The purpose of this paper is to study the range of the nonlinear operator

$$u \mapsto L_k u + G(u)\,.$$

It means that we shall investigate the following *problem*:

for given $f \in E$, find $u \in E$ such that

$$L_k u + G(u) = f\,. \tag{2.1}$$

To study (2.1) we use, as in [3] and [4], the global Lyapunov-Schmidt method. Let us denote by P the L^2-orthogonal projection of E on V and set $Q = I - P$, where I is the identity on E. Applying P and Q to (2.1) we obtain the bifurcation system

$$L_k w + QG(v + w) = Qf\,, \tag{2.2}$$

$$PG(v + w) = Pf\,. \tag{2.3}$$

It is evident that this system is equivalent to the problem (2.1).

The following lemma deals with (2.2) and it is stated without proof here.

Lemma 2.1. *Under the assumptions stated above and for fixed $f \in E$ the equation (2.2) has for each $v \in V$ precisely one solution $w(v) \in V^{\perp}$. The function $v \mapsto w(v)$ is a C^1 function of v and there exists $k > 0$ such that $\|w(v)\| \leqq k$ for all $v \in V$.*

The outline of the proof of this lemma is given in [3], a precise proof is given in [4].

3. EXISTENCE RESULT

We shall prove in this section that the range of L_k is contained in the range of the perturbed operator $L_k + G$.

Let us introduce the following function $H : V \to \mathbb{R}$ defined by

$$H(v) = 1/2(L_k\, w(v), w(v))_m + \int_\Omega \mathrm{d}x \int_0^{v + w(v)} g(x, s)\,\mathrm{d}s - (f, w(v))_m\,. \tag{3.1}$$

Theorem 3.1. *Assume* (g1)–(g3) *and let $f \in V$ be given. Then there exists at least one $u \in E$ such that* (2.1) *holds.*

Proof. Let us suppose for a moment that the function $H(\cdot)$ has at least one critical point v_0. According to the assertion of Lemma 2.1 the function $H(\cdot)$ is of class C^1, i.e.

$$(H'(v_0), h)_m = 0 \quad \text{for all} \quad h \in V.$$

By an elementary calculation we obtain from (3.1) that

$$1/2\,(L_k\, w'(v_0)\, h, w(v_0))_m + 1/2\,(L_k\, w(v_0), w'(v_0)\, h)_m + \int_\Omega g(x, w(v_0) + v_0)\, h\, \mathrm{d}x +$$
$$+ \int_\Omega g(x, w(v_0) + v_0)\, w'(v_0)\, h\, \mathrm{d}x - (f, w'(v_0)\, h)_m = 0$$

holds for each $h \in V$ (the symbol $w'(v_0)\, h$ denotes the value of the linear functional $w'(v_0)$ on h). Using the symmetry of L_k and the equation (2.2) we obtain

$$\int_\Omega g(x, v_0 + w(v_0))\, h\, \mathrm{d}x = 0$$

for all $h \in V$, which is nothing else than (2.3). The equivalence of (2.1) with (2.2), (2.3) implies that $u_0 = v_0 + w(v_0)$ is the solution of (2.1) and the theorem is proved.

Q.E.D.

It remains to prove, now, that $H(\cdot)$ has at least one critical point in V. To prove this assertion we need the following series of lemmas.

Lemma 3.2. *We have*

$$\lim_{l \to +\infty} \operatorname{meas} \{x \in \Omega;\ |w(v)(x)| \geqq l\} = 0$$

uniformly with respect to $v \in V$.

This lemma is an immediate consequence of the inequality $\|w(v)\| \leqq k$ for all $v \in V$ (see Lemma 2.1).

Lemma 3.3. *For each $l \in \mathbb{N}$ we have*

$$\lim_{\|v\| \to \infty, v \in V} \operatorname{meas} \{x \in \Omega;\ |v(x)| \leqq l\} = 0\,.$$

Proof. Suppose on the contrary that there exist $l_0 \in \mathbb{N}$, $v_n \in V$, $\|v_n\| \to \infty$ such that

$$\operatorname{meas} \{x \in \Omega;\ |v_n(x)| \leqq l_0\} \geqq \varepsilon_0 > 0\,.$$

Put $\hat{v}_n = v_n/\|v_n\|$. Then we have

$$\operatorname{meas} \{x \in \Omega;\ |\hat{v}_n(x)| \leqq l_0/\|v_n\|\} \geqq \varepsilon_0\,. \tag{3.2}$$

Since $\dim V < +\infty$ we can suppose that $\hat{v}_n \to v_0 \in V$ in $L^2(\Omega)$. By Jegorov's theorem for each $\eta > 0$ there exists $\Omega' \subset \Omega$, $\operatorname{meas} \Omega' < \eta$ and $\hat{v}_n \rightrightarrows v_0$ (uniformly) on $\Omega \setminus \Omega'$.

If we put $\eta = \varepsilon_0/2$ and take the limit for $n \to \infty$ in (3.2), we obtain

$$\text{meas}\,\{x \in \Omega;\ |v_0(x)| \leqq 0\} \geqq \varepsilon_0/2 > 0\,,$$

which is a contradiction with the unique continuation property of V (see Section 2). Q.E.D.

It is well known that the restriction $L_k \mid V^\perp : V^\perp \to V^\perp$ is an algebraic isomorphism. Its invers $(L_k \mid V^\perp)^{-1}$ will be denoted by K. Applying K on both sides of (2.2) we obtain (taking into account that $f \in V^\perp$):

$$w + KQG(v + w) = Kf\,.$$

Lemma 3.4. *If* $\int_\Omega \mathrm{d}x \int_0^{+\infty} g(x, s)\,\mathrm{d}s \in \mathbb{R}$ *then*

$$\lim_{\|v\| \to \infty} \|w(v) - Kf\| = 0 \quad \textit{and} \quad \lim_{\|v\| \to \infty} \|L_k\, w(v) - f\| = 0\,.$$

Proof. Using the Hölder inequality we obtain

$$\|w(v) - Kf\|^2 \leqq \|K\|^2 \int_\Omega |g(x, v + w(v))|^2\,\mathrm{d}x$$

and

$$\|L_k\, w(v) - f\|^2 \leqq \int_\Omega |g(x, v + w(v))|^2\,\mathrm{d}x\,.$$

Choose $\varepsilon > 0$. According to the assumptions of this lemma, (g1) and the properties of the Lebesgue integral we have $\int_0^{+\infty} \mathrm{d}s \int_\Omega |g(x, s)|^2\,\mathrm{d}x < \infty$. Hence (cf. the boundedness of $g'_s(x, s)$ in (g3)) there exists $k > 0$ such that

$$\sup_{|s| \geqq k} \int_\Omega |g(x, s)|^2\,\mathrm{d}x < \varepsilon/2\,. \tag{3.3}$$

According to Lemmas 3.2 and 3.3 we obtain the existence of such $\varkappa > 0$ that for $\|v\| \geqq \varkappa$, $v \in V$, we have

$$\text{meas}\,\Omega_k = \text{meas}\,\{x \in \Omega;\ |v(x) + w(v)\,(x)| \leqq k\} < \varepsilon/2M^2\,. \tag{3.4}$$

Using (3.3) and (3.4) we obtain for all $v \in V$, $\|v\| \geqq \varkappa$:

$$\int_\Omega |g(x, v + w(v))|^2\,\mathrm{d}x \leqq \int_{\Omega_k} |g(x, v + w(v))|^2\,\mathrm{d}x + \int_{\Omega \setminus \Omega_k} |g(x, v + w(v))|^2\,\mathrm{d}x \leqq$$

$$\leqq \int_{\Omega_k} M^2\,\mathrm{d}x + \sup_{|s| \geqq k} \int_\Omega |g(x, s)|^2\,\mathrm{d}x < \varepsilon\,. \qquad \text{Q.E.D.}$$

Lemma 3.5. *We have*

(i) $\lim_{\|v\| \to \infty} H(v) = -1/2(f, Kf)_m + \int_\Omega \mathrm{d}x \int_0^{+\infty} g(x, s)\,\mathrm{d}s$

provided $\int_\Omega \mathrm{d}x \int_0^{+\infty} g(x, s)\, \mathrm{d}s \in \mathbb{R}$;

(ii) $\lim_{\|v\| \to \infty} H(v) = \pm\infty$ *if* $\int_\Omega \mathrm{d}x \int_0^{+\infty} g(x, s)\, \mathrm{d}s = \pm\infty$.

Proof. (i) Let $\int_\Omega \mathrm{d}x \int_0^{+\infty} g(x, s)\, \mathrm{d}s \in \mathbb{R}$. According to Lemma 3.4 we have

$$\lim_{\|v\| \to \infty} [1/2(L_k w(v), w(v))_m - (f, w(v))_m] = -1/2(f, Kf)_m .$$

Let $\varepsilon > 0$ be arbitrary but fixed. We have $x \mapsto \int_0^{+\infty} |g(x, s)|\, \mathrm{d}s \in L^1(\Omega)$. Hence the functions $g_n(x) = \int_0^n g(x, s)\, \mathrm{d}s$ are uniformly integrable over Ω, $g_n(\cdot) \to \int_0^{\pm\infty} g(\cdot, s)\,.$ $.\, \mathrm{d}s$ a.e. in Ω if $n \to \pm\infty$. Using Vitali's theorem we obtain the existence of such $k \in \mathbb{N}$ that

$$\int_\Omega \left| \int_0^{\pm n} g(x, s)\, \mathrm{d}s - \int_0^{\pm\infty} g(x, s)\, \mathrm{d}s \right| \mathrm{d}x < \varepsilon/3 , \tag{3.5}$$

for all $n \geqq k$. Applying Lemma 3.3 we can choose such a $\varkappa > 0$ that

$$\int_{\Omega_k} \mathrm{d}x \int_0^{+\infty} |g(x, s)|\, \mathrm{d}s < \varepsilon/3 \quad \text{and} \quad \int_{\Omega_k} \mathrm{d}x \int_{-\infty}^0 |g(x, s)|\, \mathrm{d}s < \varepsilon/3 \tag{3.6}$$

for all $v \in V$, $\|v\| \geqq \varkappa$ (for the definition of Ω_k see the proof of Lemma 3.4). Using (g2), (3.5) and (3.6) we obtain

$$\left| \int_\Omega \mathrm{d}x \int_0^{v + w(v)} g(x, s)\, \mathrm{d}s - \int_\Omega \mathrm{d}x \int_0^{+\infty} g(x, s)\, \mathrm{d}s \right| \leqq$$
$$\leqq \int_{\Omega \setminus \Omega_k} \left| \int_0^{v + w(v)} g(x, s)\, \mathrm{d}s - \int_0^{+\infty} g(x, s)\, \mathrm{d}s \right| \mathrm{d}x +$$
$$+ \left| \int_{\Omega_k} \mathrm{d}x \int_0^{v + w(v)} g(x, s)\, \mathrm{d}s \right| + \int_{\Omega_k} \mathrm{d}x \left| \int_0^{+\infty} g(x, s)\, \mathrm{d}s \right| < \varepsilon$$

for all $\|v\| \geqq \varkappa$, which implies that

$$\lim_{\|v\| \to \infty} \int_\Omega \mathrm{d}x \int_0^{v + w(v)} g(x, s)\, \mathrm{d}s = \int_\Omega \mathrm{d}x \int_0^{+\infty} g(x, s)\, \mathrm{d}s .$$

(ii) Let $\int_\Omega \mathrm{d}x \int_0^{+\infty} g(x, s)\, \mathrm{d}s = +\infty$. Then for arbitrary $l > 0$ there exists $k > 0$ and $\eta > 0$ such that

$$\int_{\Omega \setminus \Omega'} \mathrm{d}x \int_0^{\pm n} g(x, s)\, \mathrm{d}s > l \tag{3.7}$$

for all $n \geqq k$ and $\Omega' \subset \Omega$, meas $\Omega' < \eta$.

According to Lemmas 3.2 and 3.3 we may choose a $\varkappa > 0$ such that for $v \in V$, $\|v\| \geqq \varkappa$ we have meas $\Omega_k < \eta$ and

$$\int_{\Omega_k} dx \int_0^{\pm k} |g(x, s)| \, ds < 1/l \,. \tag{3.8}$$

Using (3.7) and (3.8) we obtain

$$\int_{\Omega} dx \int_0^{v+w(v)} g(x, s) \, ds \geqq \int_{\Omega \setminus \Omega_k} dx \int_0^{v+w(v)} g(x, s) \, ds - \int_{\Omega_k} dx \int_0^{\pm k} |g(x, s)| \, |ds \geqq l - 1/l$$

for all $v \in V$, $\|v\| \geqq \varkappa$. This implies that $\lim_{\|v\| \to \infty} \int_\Omega dx \int_0^{v+w(v)} g(x, s) \, ds = +\infty$ and taking into account the boundedness of $\|w(v)\|$, we have $\lim_{\|v\| \to \infty} H(v) = +\infty$. The remaining case $(-\infty)$ is quite analogous. Q.E.D.

From the assertion of the previous lemma it immediately follows that the C^1-function $H(\cdot)$ which is defined on the finitedimensional space V must possess at least one critical point.

4. MULTIPLICITY RESULTS

In this section we shall study the properties of the range of

$$u \mapsto L_k u + G(u) \,. \tag{4.1}$$

The function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ throughout this section will be assumed to satisfy (g1)–(g3) and

$$\lim_{s \to \pm\infty} g(x, s) = 0 \tag{4.2}$$

uniformly for almost all $x \in \Omega$.

Theorem 4.1. *The range of* (4.1) *is closed.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be any sequence of elements from the range of (4.1) such that $f_n \to f$ in E. We shall prove that there exists $u \in E$ such that $L_k u + G(u) = f$. By Lemma 2.1 there exists $\{v_n\}_{n=1}^{\infty} \subset V$ such that $PG(v_n + w(v_n)) = Pf_n$, where $w(v_n) \in V^{\perp}$ are uniformly bounded solutions of the first bifurcation equation (2.2) with the right hand side Qf_n. If $Pf_n \to 0$ in V then $Pf = 0$, i.e. $f \in V^{\perp}$ and the assertion follows from Theorem 3.1. If $\|Pf_n\| \geqq$ const > 0 then according to (4.2) and Lemmas 3.2, 3.3 the sequence $\{v_n\}_{n=1}^{\infty} \subset V$ is bounded in the norm $\|\cdot\|$. Hence $u_n = v_n + w(v_n)$ is a bounded sequence in E and, possibly after passing to a suitable subsequence we can suppose that $u_n \rightharpoonup u_0$ (weakly) in E. In virtue of the compact imbedding of E into $L^2(\Omega)$ we have $G(u_n) \to G(u_0)$ strongly in E and hence $L_k u_n \to L u_0$ strongly in E. Passing to the limit in $L_k u_n + G(u_n) = f_n$ we obtain that u_0 is the solution of (2.1) with the right hand side f. Q.E.D.

The following two theorems are close to the multiplicity results [3, Th. 3.1] and [4, Th. 5.2]. Before stating the main multiplicity results we need the following lemma.

Lemma 4.2. *Let $F : \mathbb{R}^p \to \mathbb{R}$ have a continuous second Fréchet derivative which attains its local maximum at an isolated critical point $x_0 \in \mathbb{R}^p$. Then there exists an open ball $B_\varrho(x_0)$ centred at x_0 with radius $\varrho > 0$ such that*

$$\deg(F'; B_\varrho(x_0), 0) = (-1)^p .$$

The proof can be found in [1].

Theorem 4.3. *Let $f_2 \in V^\perp$ be given. Then either (1) or (2) holds, where:*

(1) the equation (2.1) with the right hand side $f = f_1 + f_2, f_1 \in V$, has no solution provided $f_1 \neq 0$ and possesses infinitely many solutions provided $f_1 = 0$; these solutions may be expressed as $u = v + w(v)$, where $w(v)$ is the solution of (2.2) with the right hand side f_2, and they form a p-dimensional C^1-manifold in E;

(2) there exists $\varepsilon(f_2) > 0$ such that (2.1) has at least one solution provided $\|f_1\| \leqq \varepsilon(f_2)$; if $0 < \|f_1\| < \varepsilon(f_2)$ then (2.1) has at least two distinct solutions.

Proof. Let us introduce the function

$$H_1(v) = H(v) - (f_1, v)_m , \quad v \in V$$

(for definition of $H(\cdot)$ see (3.1)). By an elementary calculation it is easy to see that the solutions of (2.1) with the right hand side $f = f_1 + f_2$ are in a one-to-one correspondence with the critical points of $H_1(\cdot)$. Hence if $H(\cdot) : V \to \mathbb{R}$ is a constant function on V (in this connection see Open problem 6.7), we immediately obtain the first conclusion of the theorem. Let us assume, further, that $H(\cdot)$ is not a constant. Then without loss of generality, we shall suppose that H attains its minimum at a point $v_0 \in V$. Hence there exists a ball $B_r(0)$ with centre at the origin and with a sufficiently large radius $r > 0$ such that $H(v) > H(v_0)$ for all $v \in \partial B_r(0)$. It is now obvious that there exists $\varepsilon(f_2) > 0$ such that $H_1(v) > H_1(v_0)$, for all $v \in \partial B_r(0)$, where f_1 is taken as follows: $\|f_1\| < \varepsilon(f_2)$. This fact implies that $H_1(\cdot)$ attains its minimum on $B_r(0)$ at a point $v_1 \in B_r(0)$ and so the corresponding function $u_1 = v_1 + w(v_1)$ is the solution of (2.1) with the right hand side $f = f_1 + f_2$. According to Theorem 4.1 the solution of (2.1) exists also if $\|f_1\| = \varepsilon(f_2)$. Take $0 < \|f_1\| < \varepsilon(f_2)$. Let us consider the ball $B_\varrho(v_1)$ with a sufficiently small radius ϱ. If there is another $\tilde{v}_1 \in \overline{B_\varrho(v_1)}$, $\tilde{v}_1 \neq v_1$, and $\tilde{v}_1$ is also a critical point of $H_1(\cdot)$ then $\tilde{u}_1 = \tilde{v}_1 + w(\tilde{v}_1)$ is another solution of (2.1) with the right hand side $f = f_1 + f_2$ and $\tilde{u}_1 \neq u_1$. If such a $\tilde{v}_1 \in \overline{B_\varrho(v_1)}$ does not exist we may define the Brouwer degree $\deg(\Gamma; B_\varrho(v_1), 0)$, where $\Gamma(v) = PG(v + w(v)) - f_1$ (it is easy to see that $\Gamma(v) = H_1'(v)$ and that Γ is of class C^1). According to Lemma 4.2 we have

$$\deg(\Gamma; B_\varrho(v_1), 0) \neq 0 . \tag{4.3}$$

Let us take $r > 0$ sufficiently large so that

$$\|PG(v + w(v))\| < 1/2\|f_1\| \tag{4.4}$$

for all $v \in \partial B_r(0)$. This is possible due to the assumption (4.2) and to the unique continuation property of V (see the proof of Lemma 3.4 for quite similar estimates). By virtue of (4.4) and the homotopy invariance property of the Brouwer degree it follows that

$$\deg(\tau PG(v + w(v)) - f_1; B_r(0), 0) = c$$

for all $\tau \in [0, 1]$. Since for $\tau = 0$ we obtain the Brouwer degree of a constant mapping, we have $c = 0$, and hence

$$\deg(\Gamma; B_r(0), 0) = 0 . \tag{4.5}$$

Therefore using (4.3) and (4.5) we obtain

$$\deg(\Gamma; B_r(0) \setminus \overline{B_\varrho(v_1)}, 0) \neq 0$$

and the equation $\Gamma(v) = 0$ has at least one other solution $\tilde{v}_1$ in $B_r(0) \setminus \overline{B_\varrho(v_1)}$. Then $\tilde{u}_1 = \tilde{v}_1 + w(\tilde{v}_1) \neq u_1$ is another solution of (2.1). This completes the proof. Q.E.D.

The next theorem gives a more precise information about the range of $L_k + G$ on the assumption that the nullspace of the linear part L_k is one-dimensional.

Theorem 4.4. *Let $f_2 \in V$ be given and let us suppose, moreover, that the eigenvalue λ_k has multiplicity one. Denote the corresponding eigenfunction by φ_k, $\|\varphi_k\| = 1$. Then either* (1) *or* (2) *holds, where* (1) *is the same as in Theorem* 4.3 *and*

(2) *there exist $\varepsilon_1(f_2) < 0 < \varepsilon_2(f_2)$ such that* (2.1) *with the right hand side $f = f_1 + f_2$ has at least one solution if and only if $f_1 = t\varphi_k$ is such that $t \in [\varepsilon_1(f_2), \varepsilon_2(f_2)]$; moreover, if $t \in]\varepsilon_1(f_2), 0[\cup]0, \varepsilon_2(f_2)[$ then* (2.1) *has at least two distinct solutions.*

Proof. If $H(\cdot)$ is a constant function on V we obtain again the first conclusion of the theorem. In the opposite case $H(\cdot)$ attains its maximum or minimum on V. Without loss of generality we may suppose that for $v_0 \in V$ we have

$$H(v_0) = \min_{v \in V} H(v) < \lim_{\|v\| \to \infty} H(v) . \tag{4.6}$$

Then the function $\tilde{\Gamma} : \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{\Gamma}(t) = (G(t\varphi_k + w(t\varphi_k)), \varphi_k) \quad \text{for all} \quad t \in \mathbb{R}$$

is such that $\tilde{\Gamma}(t_0) = 0$ if $t_0\varphi_k = v_0$. According to (4.2) we have $\lim_{t \to \pm\infty} \tilde{\Gamma}(t) = 0$ and using (4.6) we conclude that $\tilde{\Gamma}(t_1) > 0$ for some $t_1 > t_0$ and $\tilde{\Gamma}(t_2) < 0$ for some $t_2 < t_0$. Moreover, $\tilde{\Gamma}(\cdot)$ is a continuous function and hence using its properties stated above we have such σ_1, σ_2 that

$$\tilde{\Gamma}(\sigma_2) = \max_{t \in \mathbb{R}} \tilde{\Gamma}(t) > 0 > \min_{t \in \mathbb{R}} \tilde{\Gamma}(t) = \tilde{\Gamma}(\sigma_1).$$

Put $\tilde{\Gamma}_1(\tau) = \tilde{\Gamma}(\tau) - (t\varphi_k, \varphi_k)_m = \tilde{\Gamma}(\tau) - t$. Then if $t \in [\tilde{\Gamma}(\sigma_1), \tilde{\Gamma}(\sigma_2)]$, we obtain the existence of at least one $\tau_0 \in \mathbb{R}$ such that $\tilde{\Gamma}_1(\tau_0) = 0$ and $u_0 = \tau_0\varphi_k + w(\tau_0\varphi_k)$ is the corresponding solution of (2.1). If $t \notin [\tilde{\Gamma}(\sigma_1), \tilde{\Gamma}(\sigma_2)]$ then no such $\tau_0 \in \mathbb{R}$ does not exist. Moreover, it is easy to see that if $t \in]\tilde{\Gamma}(\sigma_1), 0[\cup]0, \tilde{\Gamma}(\sigma_2)[$, we obtain at least two distinct points $\tau_1 \neq \tau_2 \in \mathbb{R}$ such that $\tilde{\Gamma}_1(\tau_i) = 0$, $i = 1, 2$. Hence $u_i = \tau_i\varphi_k + w(\tau_i\varphi_k)$, $i = 1, 2$ are two distinct solutions of (2.1). This considerations imply that if we take $\varepsilon_1(f_2) = \tilde{\Gamma}(\sigma_1)$ and $\varepsilon_2(f_2) = \tilde{\Gamma}(\sigma_2)$, the second conclusion of the theorem is proved. Q.E.D.

5. APPLICATIONS

The results of Sections 3 and 4 may be applied, for instance, to the following types of semilinear elliptic boundary value problems:

$$\Delta u + \lambda_k u + \beta u e^{-u^2} = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega; \tag{5.1}$$

$$\Delta u + \lambda_k u + \beta e^{-u^2} \sin(u) = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega; \tag{5.2}$$

$$-\Delta^2 u + \lambda_k u + \frac{\beta u}{1 + u^8} = f \quad \text{in} \quad \Omega, \qquad u = \partial u/\partial n = 0 \quad \text{on} \quad \partial\Omega; \tag{5.3}$$

$$-\Delta^2 u + \lambda_k u + g(u) = f \quad \text{in} \quad \Omega, \qquad u = \partial u/\partial n = 0 \quad \text{on} \quad \partial\Omega, \tag{5.4}$$

where g is a bounded, odd, continuously differentiable function with compact support in $\mathbb{R}$ satisfying (g3).

We put $E = W_0^{1,2}(\Omega)$, or $E = W_0^{2,2}(\Omega)$, in the cases (5.1), (5.2), or (5.3), (5.4), respectively. The operator L_k is defined by

$$(L_k u, v)_1 = \int_\Omega \nabla u \, \nabla v \, dx - \lambda_k \int_\Omega uv \, dx$$

in the cases (5.1) and (5.2);

$$(L_k u, v)_2 = \int_\Omega \Delta u \, \Delta v \, dx - \lambda_k \int_\Omega uv \, dx$$

in the cases (5.3) and (5.4). We suppose that λ_k is any eigenvalue of the Laplace operator Δ, or the biharmonic operator Δ^2, respectively, with the Dirichlet boundary conditions. Then the operator L_k satisfies all the assumptions from Section 2. Let us note that the assumption of the unique continuation property of V is satisfied according to the result of Sitnikova [12]. The function $\beta = \beta(x)$ is measurable in x, bounded on Ω and such that (g3) is fulfilled.

6. REMARKS, OPEN PROBLEMS

Remark 6.1. Existence and multiplicity results for weakly nonlinear boundary value problems with nonlinearities presented in (5.1)−(5.4) are new and according to the author's best knowledge none of the cases (5.1)−(5.4) has been covered by any paper up to now.

Open problem 6.2. Prove or disprove:

the existence result (i.e. Theorem 3.1) remains true if the assumption (g3) is replaced by $|g'_s(x, s)| \leqq c$, for all $(x, s) \in \Omega \times \mathbb{R}$, with a constant $c > 0$ which does not depend on the spectrum of L.

Remark 6.3. An interesting approach to the study of weakly nonlinear problems is presented in [4]. The authors of this paper study the energy functional of (2.1) but to prove the existence of critical points of this functional a condition similar to (g3) is necessary.

Remark 6.4. It is obvious that the assumption (g2) can be replaced by the stronger assumption

(g2′) $g(x, s)$ is odd in s for almost all $x \in \Omega$ and

$$\int_\Omega \mathrm{d}x \int_0^{+\infty} g(x, s)\, \mathrm{d}s \in \mathbb{R} \cup \{\pm\infty\}\,.$$

All nonlinear perturbations from (5.1)−(5.1) satisfy this assumption (g2′).

Open problem 6.5. Prove or disprove:

Theorem 3.1 remains true if the assumptions (g2) (or (g2′)) and (g3) are replaced by

$$g(s)\, s \geqq 0 \quad (\text{or } g(s)\, s \leqq 0, \text{ respectively})$$

for all $s \in \mathbb{R}$.

Remark 6.6. If $\mathscr{L}$ is an elliptic differential operator of the second order and $\lambda_k = \lambda_1$ is the first simple eigenvalue with the corresponding positive eigenfunction in Ω, the problem 6.5 is solved in the affirmative in [11]; some multiplicity results are proved in [7].

Open problem 6.7. Following the proofs of Theorems 4.3 and 4.4 it is quite easy to see that if $\int_\Omega dx \int_0^{+\infty} g(x, s) \, ds = \pm\infty$ the function $H(\cdot)$ is never a constant on V and hence the situation expressed by conclusion (1) occurs for no $f_2 \in V^\perp$. The open problem is to prove the following assertion: The situation (1) from Theorems 4.3 and 4.4 occurs if and only if $g(x, s) = 0$.

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