Časopis pro pěstování matematiky

Vítězslav Novák; Miroslav Novotný

On a power of cyclically ordered sets

Časopis pro pěstování matematiky, Vol. 109 (1984), No. 4, 421--424

Persistent URL: `http://dml.cz/dmlcz/118209`

Časopis pro pěstování matematiky, roč. 109 (1984), Praha

ON A POWER OF CYCLICALLY ORDERED SETS

VÍTĚZSLAV NOVÁK, MIROSLAV NOVOTNÝ, Brno
(Received February 13, 1984)

The aim of this paper is to give a definition of an operation on the class of cyclically ordered sets – the so called power – which has a certain property analogous to that of the power of ordered sets. First, we explain the basic notions.

A ternary relation C on a set G is called a *cyclic order* ([2]) iff it is

asymmetric, i.e. $(x, y, z) \in C \Rightarrow (z, y, x) \bar{\in} C$,

cyclic, i.e. $(x, y, z) \in C \Rightarrow (y, z, x) \in C$,

transitive, i.e. $(x, y, z) \in C$, $(x, z, u) \in C \Rightarrow (x, y, u) \in C$.

A *cyclically ordered set* is a pair $\boldsymbol{G} = (G, C)$ where G is a set and C is a cyclic order on G. Note that if $\boldsymbol{G} = (G, C)$ is a cyclically ordered set and $x, y, z \in G$, $(x, y, z) \in C$, then $x \neq y \neq z \neq x$.

A cyclically ordered set $\boldsymbol{G} = (G, C)$ is *discrete* iff $C = \emptyset$; otherwise it is *nondiscrete*. A cyclically ordered set $\boldsymbol{G} = (G, C)$ is a *cycle*, iff card $G \geqq 3$ and the relation C is *linear*, i.e. $x, y, z \in G$, $x \neq y \neq z \neq x \Rightarrow$ either $(x, y, z) \in C$ or $(z, y, x) \in C$. If $\boldsymbol{G} = (G, C)$ is a cylically ordered set and $H \subseteq G$ is such a subset that the induced cyclic order $C \cap H^3$ is linear on H, then $\boldsymbol{H} = (H, C \cap H^3)$ is called a *cycle in* $\boldsymbol{G}$.

If $\boldsymbol{G} = (G, C)$ is a cyclically ordered set and $x \in G$, then the element x is called *isolated* iff there exist no $y, z \in G$ such that $(x, y, z) \in C$; otherwise it is *nonisolated*. Especially, if $\boldsymbol{G}$ is discrete, then each element of G is isolated.

Let $\boldsymbol{G} = (G, C)$, $\boldsymbol{H} = (H, D)$ be cyclically ordered sets. A mapping $f : G \to H$ is called a *homomorphism* of $\boldsymbol{G}$ into $\boldsymbol{H}$ iff it has the property

$$x, y, z \in G, \quad (x, y, z) \in C \Rightarrow (f(x), f(y), f(z)) \in D .$$

We denote by Hom $(\boldsymbol{G}, \boldsymbol{H})$ the set of all homomorphisms of $\boldsymbol{G}$ into $\boldsymbol{H}$.

Let $\boldsymbol{G} = (G, C)$, $\boldsymbol{H} = (H, D)$ be cyclically ordered sets. Put

$$\boldsymbol{G}^{\boldsymbol{H}} = (\mathrm{Hom}\,(\boldsymbol{H}, \boldsymbol{G}), T)$$

where T is a ternary relation on the set Hom $(\boldsymbol{H}, \boldsymbol{G})$ defined by

$$(f, g, h) \in T \quad \text{iff} \quad (f(x), g(x), h(x)) \in C \quad \text{for all} \quad x \in H .$$

1. Lemma. *Let G, H be cyclically ordered set. Then G^H is a cyclically ordered set.*

Proof is trivial. One can directly show that the relation T on Hom (H, G) is asymmetric, cyclic and transitive.

The cyclically ordered set G^H can be called a *cardinal power* of cyclically ordered sets G, H.

Let us denote by **3** a 3-element cycle, i.e. $\mathbf{3} = (\{0, 1, 2\}, \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\})$. One can expect – as an analogy to a cardinal power of ordered sets – that any cyclically ordered set can be isomorphically embedded into a cardinal power with base **3**. But this is not true:

2. Example. *Let G be a cyclically ordered set. Then the cardinal power $\mathbf{3}^G$ contains no 4-element cycle.*

Proof. Assume $f, g, h, k \in \text{Hom}(G, \mathbf{3})$, $(f, g, h) \in T$, $(f, h, k) \in T$. Let $x \in G$ be any element. If $f(x) = 0$, then $(f, g, h) \in T$ implies $g(x) = 1$, $h(x) = 2$ and then $(f(x), h(x), k(x)) = (0, 2, k(x))$ is not an element of the relation of **3**. Analogously we obtain a contradiction if $f(x) = 1$ and if $f(x) = 2$.

Thus, if G is a cyclically ordered set that contains a 4-element cycle, then G can be embedded into no cardinal power with base **3**. We propose another operation of a power of cyclically ordered sets which removes this defect.

In the sequel we assume that $G = (G, C)$ is a cyclically ordered set which is nondiscrete and $H = (H, D)$ is a cyclically ordered set without isolated elements. Let $\mathfrak{C}(H)$ be the set of all cycles in H. Put

$$P(G, H) = \Big(\bigcup_{X \in \mathfrak{C}(H)} \text{Hom}(X, G), R\Big)$$

where $(f, g, h) \in R$ iff $\text{dom} f = \text{dom}\, g = \text{dom}\, h$ and $(f(x), g(x), h(x)) \in C$ for any $x \in \text{dom} f$.

3. Lemma. *$P(G, H)$ is a cyclically ordered set.*

Proof is easy.

Choose an element $\omega \in G$ which is nonisolated and denote for any $x \in H$

$$U(x) = \{f \in \bigcup_{X \in \mathfrak{C}(H)} \text{Hom}(X, G);\ x \in \text{dom} f \text{ and } f(x) = \omega\}\,.$$

4. Lemma. *If $x, y \in H$, $x \neq y$, then $U(x) \cap U(y) = \emptyset$.*

Proof. Assume that there exists an $f \in U(x) \cap U(y)$. If $\text{dom} f = X$, then $X = (X, D \cap X^3)$ is a cycle in H and $x, y \in X$; simultaneously $f(x) = \omega = f(y)$. Find an element $z \in X$ such that either $(x, y, z) \in D$ or $(z, y, x) \in D$. Then either $(f(x), f(y), f(z)) \in C$ or $(f(z), f(y), f(x)) \in C$ and this is impossible, for $f(x) = f(y) = \omega$.

Define on the set $\{U(x);\ x \in H\}$ a ternary relation S by $(U(x), U(y), U(z)) \in S$ iff there exist $f \in U(x)$, $g \in U(y)$, $h \in U(z)$ with $(f, g, h) \in R$.

5. Lemma. *If* $(U(x), U(y), U(z)) \in S$, *then* $x \neq y \neq z \neq x$.

Proof. Let $(U(x), U(y), U(z)) \in S$. Then there exist $f \in U(x)$, $g \in U(y)$, $h \in U(z)$ with $(f, g, h) \in R$. Thus $\operatorname{dom} f = \operatorname{dom} g = \operatorname{dom} h = X$, where $\boldsymbol{X} = (X, D \cap X^3) \in \mathfrak{C}(\boldsymbol{H})$, $x, y, z \in X$ and $(f(t), g(t), h(t)) \in C$ for any $t \in X$. Assume $x = y$; then $f(x) = \omega = g(y) = g(x)$ so that $(f(x), g(x), h(x)) \in C$ cannot hold, which is a contradiction. Similarly, neither $x = z$ nor $y = z$ is possible.

6. Lemma. *If* $x, y, z \in H$, $(x, y, z) \in D$, *then* $(U(z), U(y), U(x)) \in S$.

Proof. Denote $X = \{x, y, z\}$, $\boldsymbol{X} = (X, D \cap X^3)$; then $\boldsymbol{X} \in \mathfrak{C}(\boldsymbol{H})$. Further, choose two elements $a, b \in G$ such that $(\omega, a, b) \in C$ and define mappings $f, g, h \colon X \to G$ by

$$f(x) = a\,,\quad f(y) = b\,,\quad f(z) = \omega\,,$$
$$g(x) = b\,,\quad g(y) = \omega\,,\quad g(z) = a\,,$$
$$h(x) = \omega\,,\quad h(y) = a\,,\quad h(z) = b\,.$$

We see easily that $f, g, h \in \operatorname{Hom}(\boldsymbol{X}, \boldsymbol{G})$, $f \in U(z)$, $g \in U(y)$, $h \in U(x)$ and $(f(t), g(t), h(t)) \in C$ for any $t \in X$. Thus $(f, g, h) \in R$ and $(U(z), U(y), U(x)) \in S$.

7. Lemma. *If* $x, y, z \in H$, $(U(x), U(y), U(z)) \in S$, *then* $(z, y, x) \in D$.

Proof. Let $(U(x), U(y), U(z)) \in S$, i.e. there exist $f \in U(x)$, $g \in U(y)$, $h \in U(z)$ with $(f, g, h) \in R$. Thus $\operatorname{dom} f = \operatorname{dom} g = \operatorname{dom} h = X$, where $\boldsymbol{X} = (X, D \cap X^3) \in \mathfrak{C}(\boldsymbol{H})$, $x, y, z \in X$ and $(f(t), g(t), h(t)) \in C$ for any $t \in X$. Further $f(x) = \omega = g(y) = h(z)$. By Lemma 5, we have $x \neq y \neq z \neq x$. As $\boldsymbol{X}$ is a cycle in $\boldsymbol{H}$, we have either $(x, y, z) \in D$ or $(z, y, x) \in D$. Assume $(x, y, z) \in D$. As $f, g, h \in \operatorname{Hom}(\boldsymbol{X}, \boldsymbol{G})$, we have $(f(x), f(y), f(z)) \in C$, i.e. $(\omega, f(y), f(z)) \in C$, and $(g(x), g(y), g(z)) \in C$, i.e. $(\omega, g(z), g(x)) \in C$, and $(h(x), h(y), h(z)) \in C$, i.e. $(\omega, h(x), h(y)) \in C$. Besides, we have $(f(x), g(x), h(x)) \in C$, i.e. $(\omega, g(x), h(x)) \in C$, and $(f(y), g(y), h(y)) \in C$, i.e. $(\omega, h(y), f(y)) \in C$, and $(f(z), g(z), h(z)) \in C$, i.e. $(\omega, f(z), g(z)) \in C$. Then, by a successive application of the transitivity of the relation C, we obtain

$$\begin{matrix}(\omega, f(y), f(z)) \in C \\ (\omega, f(z), g(z)) \in C\end{matrix} \Rightarrow (\omega, f(y), g(z)) \in C\,,$$

$$\begin{matrix}(\omega, f(y), g(z)) \in C \\ (\omega, g(z), g(x)) \in C\end{matrix} \Rightarrow (\omega, f(y), g(x)) \in C\,,$$

$$\begin{matrix}(\omega, f(y), g(x)) \in C \\ (\omega, g(x), h(x)) \in C\end{matrix} \Rightarrow (\omega, f(y), h(x)) \in C\,,$$

$$\begin{matrix}(\omega, f(y), h(x)) \in C \\ (\omega, h(x), h(y)) \in C\end{matrix} \Rightarrow (\omega, f(y), h(y)) \in C\,,$$

and this contradicts $(\omega, h(y), f(y)) \in C$. Thus, $(z, y, x) \in D$.

Now, let us put

$$P_\omega(G, H) = (\{U(x); x \in H\}, S).$$

8. Theorem. *$P_\omega(G, H)$ is a cyclically ordered set.*

Proof. Assume that there exist elements $x, y, z \in H$ such that $(U(x), U(y), U(z)) \in S$, $(U(z), U(y), U(x)) \in S$. Then by Lemma 5, we have $x \neq y \neq z \neq x$ and Lemma 7 implies $(z, y, x) \in D$, $(x, y, z) \in D$. This contradicts the asymmetry of D and hence S is asymmetric. The cyclicity of the relation S follows directly from its definition. We prove that S is transitive. Let $x, y, z, w \in H$, $(U(x), U(y), U(z)) \in S$, $(U(x), U(z), U(w)) \in S$. Then by Lemma 7, $(z, y, x) \in D$, $(w, z, x) \in D$. Hence $(x, w, z) \in D$, $(x, z, y) \in D$ and the transitivity of D yields $(x, w, y) \in D$, thus also $(w, y, x) \in D$. By Lemma 6 we have $(U(x), U(y), U(w)) \in S$ and S is transitive.

9. Theorem. *$P_\omega(G, H)$ is antiisomorphic with H.*

Proof. The mapping $U: x \to U(x)$ is clearly a surjective mapping of H onto $\{U(x); x \in H\}$; by Lemma 4, it is a bijection. Then Lemmas 6 and 7 imply that U is an antiisomorphism of H onto $P_\omega(G, H)$.

10. Corollary. *Let $G = (G, C)$ be a cyclically ordered set without isolated elements. Then the cyclically ordered set $P_0(3, G)$ is antiisomorphic with G.*

References

[1] *G. Birkhoff:* Generalized arithmetic. Duke Math. Journ. *9* (1942), 283—302.

[2] *V. Novák:* Cyclically ordered sets. Czech. Math. Journ. *32* (*107*) (1982), 460—473.

[3] *V. Novák:* Operations on cyclically ordered sets. Arch. Math., to appear.

[4] *V. Novák, M. Novotný:* On determination of a cyclic order. Czech. Math. Journ. *33* (*108*) (1983), 555—563.

[5] *M. Novotný:* On representation of partially ordered sets by sequences of zeros and ones. (Czech.) Čas. pěst. mat. *78* (1953), 61—64.

[6] *M. Novotný, V. Novák:* Dimension theory for cyclically and cocyclically ordered sets. Czech. Math. Journ. *33* (*108*) (1983), 647—653.

Authors' addresses: V. Novák, 662 95 Brno, Janáčkovo nám. 2a, (Přírodovědecká fakulta UJEP), M. Novotný, 603 00 Brno, Mendlovo nám. 1, (Matematický ústav ČSAV).