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ON THE SET OF POINTS OF DISCONTINUITY FOR FUNCTIONS WITH CLOSED GRAPHS

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For two topological spaces $X$ and $Y$ and any function $f : X \to Y$ the subset $\{(x, f(x)) ; x \in X\}$ of the space $X \times Y$ (with Tychonoff's topology) is called the graph of $f$ and is denoted by $G(f)$. We denote by $C_f(D_f)$ the set of all such points at which the function $f$ defined on $X$ is continuous (discontinuous).

I. Baggs [1] dealt with the set of points of discontinuity of functions with closed graphs. In this paper we shall generalize some results of the paper [1].

1. PRELIMINARIES

First we recall definitions and some basic properties.

**Proposition A.** Let a function $f : X \to Y$ have a closed graph. If $K$ is a compact subset of $Y$ then $f^{-1}(K)$ is a closed subset of $X$. (See [2; Theorem 3.6].)

This proposition has the following corollary.

**Proposition B.** Let a function $f : X \to Y$ have a closed graph. Then $f^{-1}(y)$ is a closed subset of $X$ for each $y \in Y$. (See [7; Theorem 1].)

**Proposition C.** Let $f : X \to Y$ be any function where $Y$ is a locally compact Hausdorff space. If for each compact $K \subset Y$, $f^{-1}(K)$ is closed, then $G(f)$ is closed. (See [9; Theorem 6].)

**Definition 1.** Let $X$ and $Y$ be topological spaces, let $f : X \to Y$ be a function and let $p \in X$. Then $f$ is said to be $c$-continuous at $p$ provided the following holds: if $U$ is an open subset of $Y$ containing $f(p)$ such that $Y - U$ is compact, then there is an open subset $V$ of $X$ containing $p$ such that $f(V) \subset U$. The function $f$ is said to be $c$-continuous (on $X$) provided $f$ is $c$-continuous at each point of $X$. (See [3; Definition 1].)
Proposition D. Let \( f : X \to Y \) be any function where \( Y \) is Hausdorff. Then the following statements are equivalent:

1. \( f \) is c-continuous, and
2. if \( K \) is a compact subset of \( Y \), then \( f^{-1}(K) \) is a closed subset of \( X \). (See [3; Theorem 1].)

Corollary 1. Let \( f : X \to Y \) be any function where \( Y \) is a locally compact Hausdorff space. Then \( f \) is c-continuous if and only if \( G(f) \) is closed.

Definition 2. A function \( f : X \to Y \) is locally bounded at \( x_0 \in X \) if and only if there exists a compact subset \( K \) of \( Y \) such that \( x_0 \in \text{Int}(f^{-1}(K)) \). We denote by \( B_f \) the set of all such points at which the function \( f \) is locally bounded.

Lemma A. Let \( f : X \to Y \) be given. Then \( G(f) \) is closed if and only if for each \( x \in X \) and \( y \in Y \), where \( y \neq f(x) \), there exist open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap V = \emptyset \). (See [8; Lemma].)

Theorem 1. Let \( f : X \to Y \) be given. If \( G(f) \) is closed, then

\[ B_f \subseteq C_f. \]

Proof. We may assume that \( Y \) has at least two elements (in the opposite case we evidently have \( B_f = C_f \)). Let the set \( G(f) \) be closed. Let \( x_0 \in B_f \). By Definition 2 there exists a compact set \( K \) (in \( Y \)) such that \( x_0 \in \text{Int}(f^{-1}(K)) \). Let \( V \) be an open neighbourhood of the point \( f(x_0) \). Since \( K - V \) is compact and \( G(f) \) is closed, \( f^{-1}(K - V) \) is closed by Proposition A. Put

\[ U = \text{Int}(f^{-1}(K)) - f^{-1}(K - V). \]

Evidently \( U \) is an open neighbourhood of the point \( x_0 \). We shall prove that \( f(U) \subseteq V \). Let \( x \in U \). Since \( f(x) \in K \) and \( f(x) \notin K - V \), evidently \( f(x) \in V \). Hence \( x_0 \in C_f \).

Corollary 2. Let \( f : X \to Y \) be any function where \( Y \) is a locally compact space. If \( G(f) \) is closed, then \( B_f \subseteq C_f \).

The converse to Corollary 2 is not necessarily true as the following example shows.

Example 1. Let \( X = Y = \mathbb{R} \) (where \( \mathbb{R} \) denotes the set of all real numbers) with the usual topology. Define a function \( f : X \to Y \) as follows:

\[ f(x) = \begin{cases} 
\frac{1}{x} \sin \frac{1}{x} & \text{for } x \neq 0, \\
0 & \text{for } x = 0.
\end{cases} \]

Then \( G(f) \) is not closed, but \( Y \) is locally compact and \( B_f = C_f \).
Lemma 1. Let the function \( f : X \to Y \) have a closed graph. If \( K \) is a compact subset of \( Y \) then \( f^{-1}(K) - B_f \) is nowhere dense in \( X \).

Proof. Let \( K \) be a compact subset of \( Y \). Put \( A = f^{-1}(K) - B_f \).

By Definition 2 it is easy to see that \( B_f \) is open. By Proposition A the set \( f^{-1}(K) \) is closed. Therefore \( A \) is closed. Now we shall prove that \( \text{Int}(A) = \emptyset \). Let \( x \in \text{Int}(A) \). Then \( A \) is a neighbourhood of the point \( x \) such that \( f(A) \subseteq K \). Since \( K \) is compact by Definition 2 we have \( x \in B_f \). This leads to a contradiction because \( x \in A \subseteq X - B_f \). Therefore \( A \) is nowhere dense in \( X \).

Theorem 2. Let \( f : X \to Y \) be any function where \( Y \) is a \( \sigma \)-compact space (i.e. \( Y \) is the countable union of compact sets). If \( G(f) \) is closed, then \( X - B_f \) is closed and of the first category (in \( X \)).

Proof. By the assumption, \( Y = \bigcup_{n=1}^{\infty} K_n \), where each \( K_n \) is compact. Let \( n \in N \) (where \( N \) denotes the set of all positive integers). Put \( A_n = f^{-1}(K_n) - B_f \).

By Lemma 1 the set \( A_n \) is nowhere dense in \( X \). Hence \( X - B_f = \bigcup_{n=1}^{\infty} A_n \) is of the first category in \( X \).

2. REAL FUNCTIONS WITH CLOSED GRAPHS

Let \( X \) be a topological space. Denote by \( U(X) \) the class of all real functions defined on \( X \) with closed graphs.

From Corollary 2, Lemma 1, Proposition B and Theorem 2 we obtain the following three theorems.

Theorem 3. Let \( f \in U(X) \). Then the set \( D_f \) is closed and of the first category (in \( X \)).

Theorem 4. Let \( f \in U(X) \). Then \( D_f \cap f^{-1}(0) \) is closed and nowhere dense (in \( X \)).

Theorem A. Let \( X \) be a \( T_2 \) Baire space. If \( f : X \to \mathbb{R}^n \) (\( \mathbb{R}^n \) — the Euclidean \( n \)-space) has a closed graph, then \( D_f \) is closed and nowhere dense in \( X \). (See [1], and for metric spaces see [5; Theorems 4 and 5].)

Theorem B. A set \( F \subseteq \mathbb{R} \) is closed and nowhere dense if and only if there exists a function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f \) has a closed graph and \( D_f = F \). (See [1].)
Theorem 5. Let $F$ be a closed, $G_6$ and nowhere dense subset of a normal topological space $X$. Let $u : X \to (0, 1)$ be a continuous function such that $u^{-1}(0) = F$. Define a function $g : X \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} 1/u(x) & \text{for } x \in X - F, \\ 0 & \text{for } x \in F. \end{cases}$$

Then $g$ has a closed graph and $D_g = F$.

Proof. We show that the graph of $g$ is closed. Let $\{(x_a, g(x_a))\}_{a \in A}$ be a convergent net of points of the graph of $g$, i.e. there exist $x_0$ and $y_0$ such that $(x_a, g(x_a)) \to (x_0, y_0) \in X \times \mathbb{R}$. We distinguish two cases.

a. Let there exist $x_0$ such that for every $a > a_0$ we have $x_a \in F$. Since $x_a \to x_0$ and $F$ is closed, we obtain $x_0 \in F$. Hence $g(x_0) = 0 = y_0$.

b. For each $a$ let there exist $\beta > \alpha$ such that $x_\beta \notin F$. It follows from the definition of $g$ that $g(x) \geq 1$ whenever $x \in X - F$. The convergence of the net $\{g(x_a)\}_{a \in A}$ implies that there is $x_0$ such that for every $a > a_0$ we have $x_a \in X - F$. Since $u$ is continuous at the point $x_0$ and $g(x_\beta) \to y_0$, we obtain $x_0 \in X - F$. Since $g$ is continuous on the set $X - F$, it is not difficult to verify that $g(x_\beta) \to g(x_0)$. Hence $g(x_0) = y_0$.

Finally, we show that $D_g = F$. Evidently $g$ is continuous on the set $X - F$. Let $x \in F$. Because the set $F$ is nowhere dense, we have $\omega_g(x) \geq 1$ for the oscillation of $g$ in $x$. Hence $x \in D_g$. The following example shows that there exists a metric space $X$ and a function $f \in U(X)$ such that $D_f$ is not nowhere dense.

Example 2. Let $X = \{x_1, x_2, \ldots\}$ be a countably dense subset of $\mathbb{R}$. Define a function $f : X \to \mathbb{R}$ as follows:

$$f(x_n) = n \quad (n = 1, 2, \ldots).$$

Then $f$ has a closed graph, but $D_f = X$ is not nowhere dense in $X$.

Proposition 1. Let $X$ be a topological space. Let $f \in U(X)$. Then $|f| \in U(X)$.

Proof. Let $x_0 \in X$. Let $y \neq |f(x_0)|$. First suppose that $y \geq 0$. Since $y \neq f(x_0)$, by Lemma A there exist $\delta_1 > 0$ and a neighbourhood $U_1$ of the point $x_0$ such that

$$f(U_1) \cap (y - \delta_1, y + \delta_1) = \emptyset.$$ 

Since $-y \neq f(x_0)$, by Lemma A there exist $\delta_2 > 0$ and neighbourhood $U_2$ of the point $x_0$ such that

$$f(U_2) \cap (-y - \delta_2, -y + \delta_2) = \emptyset.$$ 

Put

$$U = U_1 \cap U_2,$$

$$\delta = \min(\delta_1, \delta_2),$$

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Let \( V = (y - \delta, y + \delta) \).

Let \( x \in U \). If \( f(x) \geq 0 \), since \( f(x) \notin (y - \delta_1, y + \delta_1) \), we have \(|f(x)| - y| = |f(x) - y| \geq \delta \geq \delta_1 \). Therefore \( |f(x)| \notin V \). If \( f(x) < 0 \), then \( f(x) \notin (-y - \delta_2, -y + \delta_2) \), hence \( |y - f(x)| = |y + f(x)| \geq \delta_2 \geq \delta \). Therefore \( |f(x)| \notin V \). In the case \( y < 0 \) put \( U = X, V = (-\infty; 0) \). Then by Lemma A the function \( f \) has a closed graph.

**Proposition 2.** Let \( X \) be a topological space. Let \( \alpha \) be a real number. Let \( f \in U(X) \).

Then \( \alpha \cdot f \in U(X) \).

**Proof.** It is obvious that for \( \alpha = 0 \) we have \( \alpha \cdot f \in U(X) \). Suppose that \( \alpha \neq 0 \).

Let \( x_0 \in X \). Let \( K \) be a compact subset of \( R \) such that \( \alpha \cdot f(x_0) \notin K \). Since \( K \) is closed, there exists \( \varepsilon > 0 \) such that

\[
(\alpha \cdot f(x_0) - \varepsilon, \alpha \cdot f(x_0) + \varepsilon) \cap K = \emptyset.
\]

Let \( k > 0 \) be a bound of the set \( K \) (i.e., \( K \subset (-k, k) \)). Put

\[
h = \max(k, k/|\alpha|),
\]

\[
K_1 = (-h, h) - (f(x_0) - \varepsilon/|\alpha|, f(x_0) + \varepsilon/|\alpha|).
\]

Since \( f(x_0) \notin K_1 \) and \( K_1 \) is compact, there exists a neighbourhood \( U \) of the point \( x_0 \) such that

\[
f(U) \cap K_1 = \emptyset.
\]

Let \( x \in U \). If \( \alpha \cdot f(x) \notin (-k, k) \), evidently \( \alpha \cdot f(x) \notin K \). Let \( \alpha \cdot f(x) \in (-k, k) \). Then \( f(x) \in (-h, h) \), therefore by (4) we have \( f(x) \in (f(x_0) - \varepsilon/|\alpha|, f(x_0) + \varepsilon/|\alpha|) \). Thus \( |\alpha \cdot f(x) - \alpha \cdot f(x_0)| = |\alpha| \cdot |f(x) - f(x_0)| < |\alpha| \cdot \varepsilon/|\alpha| = \varepsilon \), hence by (3) we have \( \alpha \cdot f(x) \notin K \). Then Corollary 1 yields \( \alpha \cdot f \in U(X) \).

**Remark 1.** Propositions 1 and 2 are proved in the paper [6] for \( X \) a metric space.

It is known that the class \( U(X) \) is not closed with respect to addition (see [6; Example 3]). We prove that \( U(X) \) is closed with respect to addition of nonnegative functions.

**Theorem 6.** Let \( X \) be a topological space. Let \( f, g \in U(X) \) be nonnegative functions.

Then \( f + g \in U(X) \).

**Proof.** Let \( x_0 \in X \). Let \( K \) be a compact subset of \( R \) such that \( f(x_0) + g(x_0) \notin K \).

The closedness of the set \( K \) implies that there exists \( \varepsilon > 0 \) such that

\[
(f(x_0) + g(x_0) - \varepsilon, f(x_0) + g(x_0) + \varepsilon) \cap K = \emptyset.
\]

Let \( k > 0 \) be a bound of the set \( K \) (i.e., \( K \subset (-k, k) \)). Put

\[
K_1 = (0, k) - (f(x_0) - \varepsilon/2, f(x_0) + \varepsilon/2),
\]

\[
K_2 = (0, k) - (g(x_0) - \varepsilon/2, g(x_0) + \varepsilon/2),
\]

\[
K_3 = (0, k) - (f(x_0) + g(x_0) - \varepsilon, f(x_0) + g(x_0) + \varepsilon).
\]

Therefore \( f + g \notin K_1 \) in the case \( f(x_0) + g(x_0) < 0 \), and \( f + g \notin K_2 \) in the case \( f(x_0) + g(x_0) > 0 \). Also \( f + g \notin K_3 \). Therefore \( f + g \notin \emptyset \) and \( f + g \in U(X) \).
\[ K_2 = \langle 0, k \rangle - (g(x_0) - \varepsilon/2, g(x_0) + \varepsilon/2). \]

Since \( f \in U(X) \), by Corollary 1 there exists a neighbourhood \( U_1 \) of the point \( x_0 \) such that
\[ f(U_1) \cap K_1 = \emptyset. \]

Since \( g \in U(X) \), by Corollary 1 there exists a neighbourhood \( U_2 \) of the point \( x_0 \) such that
\[ g(U_2) \cap K_2 = \emptyset. \]

Put
\[ U = U_1 \cap U_2. \]

Let \( x \in U \). If \( f(x) + g(x) > k \), evidently \( f(x) + g(x) \notin K \). Let \( f(x) + g(x) \in \langle 0, k \rangle \).

Since by (6) we have \( f(x) \in \langle 0, k \rangle - K_1 \), by the definition of \( K_1 \) we obtain
\[ f(x) \in (f(x_0) - \varepsilon/2, f(x_0) + \varepsilon/2). \]

Since by (7) we have \( g(x) \in \langle 0, k \rangle - K_2 \), by the definition of \( K_2 \) we obtain
\[ g(x) \in (g(x_0) - \varepsilon/2, g(x_0) + \varepsilon/2). \]

From (8) and (9) it follows that \(|f(x) + g(x)) - (f(x_0) + g(x_0))| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \varepsilon/2 + \varepsilon/2 = \varepsilon\), hence by (5) we have \( f(x) + g(x) \notin K \). Therefore \((f + g)(U) \cap K = \emptyset. \) By Corollary 1 we obtain \( f + g \in U(X) \).

**Corollary 3.** Let \( X \) be a topological space. Let \( f, g \in U(X) \). Then \(|f| + |g| \in U(X)\).

**Definition 3.** A topological space \( X \) is called **perfectly normal** if and only if it is normal and each closed subset of \( X \) is \( G_d \). (See [4], p. 181.)

**Theorem 7.** Let \( X \) be a perfectly normal topological space. Then \( A \subset X \) is closed and of the first category in \( X \) if and only if there exists a function \( f \in U(X) \) such that
\[ D_f = A. \]

**Proof.** Necessity follows from Theorem 3. Sufficiency. Let \( A \subset X \) be closed and of the first category in \( X \). Then \( A = \bigcup_{n=1}^{\infty} A_n \), where each \( A_n \) is a closed nowhere dense subset of \( X \), \( A_n \subset A_{n+1} \) \((n = 1, 2, \ldots) \). Let \( g : X \to \langle 0, 1 \rangle \) be a continuous function such that \( g^{-1}(0) = A \). Let \( g_n : X \to \langle 0, 1 \rangle \) \((n = 1, 2, \ldots) \) be continuous functions such that for each \( n \in \mathbb{N} \)
\[ g_n^{-1}(0) = A, \]
\[ g_n(x) \geq g(x) \text{ for each } x \in X. \]

The existence of functions \( g, g_n \) \((n = 1, 2, \ldots) \) follows from Urysohn’s lemma. For each \( n \in \mathbb{N} \) define a function \( f_n : X \to \mathbb{R} \) as follows:
\[
 f_n(x) = \begin{cases} 
 \frac{1}{g_n(x)} & \text{for } x \in X - A_n, \\
 0 & \text{for } x \in A_n.
\end{cases}
\]

By Theorem 5 we have \( f_n \in U(X) \) \( (n = 1, 2, \ldots) \). Consider the series
\[
(12) \quad \sum_{n=1}^{\infty} (1/2^n) \cdot f_n.
\]

We now show that the series (12) is convergent to some function \( f : X \to R \). If \( x \in A_m \) for some \( m \in N \), then
\[
0 \leq \sum_{n=1}^{\infty} ((1/2^n) \cdot f_n(x)) = \sum_{n=1}^{m} ((1/2^n) \cdot f_n(x)) < +\infty.
\]

If \( x \in X - A \), then
\[
0 \leq \sum_{n=1}^{\infty} ((1/2^n) \cdot f_n(x)) = \sum_{n=1}^{\infty} ((1/2^n) \cdot (1/g_n(x))) \leq \\
\leq \sum_{n=1}^{\infty} ((1/2^n) \cdot (1/g(x))) = 1/g(x) < +\infty.
\]

We now show that \( D_f = A \). First we shall prove that \( X - A \subset C_f \). Let \( b \in X - A \). Since \( g(b) > 0 \) and \( g \) is continuous at the point \( b \), there exists a neighbourhood \( U \) of the point \( b \) such that
\[
(13) \quad \forall x \in U : g(x) > g(b)/2.
\]

Evidently \( U \subset X - A \). Hence by (13) we have for each \( x \in U \)
\[
f_n(x) = 1/g_n(x) \leq 1/g(x) < 2|g(b)\ (n = 1, 2, \ldots).
\]

Therefore the series (12) is uniformly convergent on the set \( U \). Since all functions \( f_n \) are continuous on \( U \), the function \( f \) is continuous at the point \( b \). Now we show that \( A \subset D_f \). Let \( a \in A \). Then \( a \in A_m \) for some \( m \in N \). We shall prove that for each neighbourhood \( V \) of the point \( a \) and for each \( n \in N \) there exists a point \( y \in V \) such that \( f(y) > n \). Let \( V \) be a neighbourhood of the point \( a \). Let \( n \in N \). Since \( g_m \) is continuous at the point \( a \), there exists a neighbourhood \( W \) of the point \( a \) such that
\[
(14) \quad \forall x \in W : g_m(x) < 2^{-m}/n.
\]

Since \( A_m \) is closed and nowhere dense, there exists a point \( y \in V \cap W \) such that \( y \in X - A_m \). Hence by (14) we have
\[
f(y) \geq (1/2^n) \cdot (1/g_m(y)) > n.
\]

Therefore \( a \in X - B_f = D_f \).

Now we shall prove that \( f \in U(X) \). Let \( K \) be a compact subset of \( R \). We now show that \( X - f^{-1}(K) \) is open. Let \( x_1 \in A_m - f^{-1}(K) \) for some \( m \in N \). Put
\[ h = \sum_{n=1}^{m} \left( \frac{1}{2^n} \cdot f_n \right). \]

By Theorem 5, Proposition 2 and Theorem 6 we obtain \( h \in U(X) \). Since \( f(x_1) \notin K \) and \( K \) is closed, there exists \( \varepsilon > 0 \) such that
\[ (f(x_1) - \varepsilon, f(x_1) + \varepsilon) \cap K = \emptyset. \]

Let \( k > 0 \) be a bound of the set \( K \) (i.e. \( K \subset \langle -k, k \rangle \)). Put
\[ K_1 = \langle 0, k \rangle - (f(x_1) - \varepsilon, f(x_1) + \varepsilon). \]

Since \( h(x_1) = f(x_1) \notin K_1 \), \( K_1 \) is compact and \( h \in U(X) \), by Proposition A the set \( X - h^{-1}(K_1) \) is an open neighbourhood of the point \( x_1 \). Since \( g_m \) is continuous at the point \( x_1 \), there exists a neighbourhood \( U_1 \) of the point \( x_1 \) such that \( U_1 \subset X - h^{-1}(K_1) \) and for each \( x \in U_1 \) we have
\[ (15) \quad g_m(x) < 2^{-m}/(f(x_1) + \varepsilon). \]

If \( x \in U_1 \cap A_m \), then \( f(x) = h(x) \notin K_1 \). Therefore \( f(x) \notin K \). If \( x \in U_1 - A_m \), then by (15) we have
\[ h(x) \geq (1/2^m) \cdot f_m(x) = (1/2^m) \cdot (1/g_m(x)) > f(x_1) + \varepsilon. \]

Since \( h(x) \notin K_1 \), we obtain \( h(x) \notin \langle 0, k \rangle \). Hence \( f(x) \geq h(x) > k \), then \( f(x) \notin K \). Therefore the point \( x_1 \) has a neighbourhood \( U_1 \) such that \( U_1 \subset X - f^{-1}(K) \).

Let \( x_2 \in (X - A) - f^{-1}(K) \). Since \( x_2 \in C_f \), the set
\[ U_2 = X - f^{-1}(K) = f^{-1}(R - K) \]
is a neighbourhood of the point \( x_2 \).

Therefore the set \( X - f^{-1}(K) \) is open. By Proposition C we have \( f \in U(X) \).

This theorem has the following corollary.

**Theorem C.** Let \( X \) be a Baire metric space. Then \( F \subset X \) is closed and nowhere dense in \( X \) if and only if there exists a function \( f \in U(X) \) such that \( D_f = F \).

The following example shows that the assumption "\( X \) is perfectly normal" in Theorem 7 cannot be replaced by the assumption "\( X \) is normal".

**Example 3.** Let \( X = \{ \omega; \omega \leq \Omega \} \) (where \( \Omega \) denotes the first uncountable ordinal number) with the order topology. It is well known that \( X \) is a normal space, and the set \( \{ \Omega \} \) is closed and nowhere dense in \( X \) but for each \( f \in U(X) \) we have \( D_f \neq \{ \Omega \} \).

(See [1].)

**References**


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