Gary Chartrand; Farrokh Saba; Hung Bin Zou
Edge rotations and distance between graphs

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INTRODUCTION

In [1] Zelinka introduced the following definition of distance between two graphs of the same order. Let $G_1$ and $G_2$ be two graphs of order $p$. Then the distance $\delta(G_1, G_2)$ between $G_1$ and $G_2$ is $n$ ($0 \leq n \leq p - 1$) if $p - n$ is the order of a largest graph that is an induced subgraph of both $G_1$ and $G_2$.

Zelinka showed that on the family of graphs having a fixed order, the above distance function $\delta$ produces a metric space. He further showed for graphs $G_1$ and $G_2$ of order $p$ that $\delta(G_1, G_2) \leq p - 1$ and $\delta(G_1, G_2) = \delta(\overline{G}_1, \overline{G}_2)$, where $\overline{G}$ denotes the complement of $G$.

In this paper, we introduce a new distance function defined on graphs having the same order and the same size (number of edges).

EDGE ROTATIONS AND TRANSFORMATIONS

We say that a graph $G$ can be transformed into a graph $H$ by an edge rotation if $G$ contains distinct vertices $u$, $v$ and $w$ such that $uv \in E(G)$, $uw \notin E(G)$ and $H \cong G - uv + uw$. In this case, $G$ is transformed into $H$ by "rotating" the edge $uv$ of $G$ into $uw$. Observe that a graph $G$ can be transformed into some graph $H$ by an edge rotation if and only if $G$ is neither complete nor empty.

Figure 1 shows graphs $G$, $H_1$ and $H_2$. Note that $G$ can be transformed into $H_1$ by an edge rotation ($xy$ is rotated into $xz$). Also, $G$ can be transformed into $H_2$ by an edge rotation ($xw$ is rotated into $xz$). Further observe that $G \not\cong H_1$ and $G \cong H_2$.

![Figure 1](image_url)
It is immediate that a graph $G$ can be transformed into a graph $H$ by an edge rotation if and only if $H$ can be transformed into $G$ by an edge rotation. More generally, we say simply that $G_1$ can be transformed into $G_2$, written $G_1 \rightarrow G_2$, if either (1) $G_1 \cong G_2$, or (2) there exists a sequence

$$G_1 \cong H_0, H_1, \ldots, H_n \cong G_2 \ (n \geq 1)$$

of graphs such that $H_i$ can be transformed into $H_{i+1}$ by an edge rotation for $i = 0, 1, \ldots, n - 1$. It is obvious that the relation “can be transformed into” is an equivalence relation on the set of all graphs. Moreover, if $G_1$ and $G_2$ are graphs for which $G_1 \rightarrow G_2$, then clearly $G_1$ and $G_2$ have the same order (the same number of vertices) and the same size (the same number of edges). It is perhaps less clear that the converse of the preceding implication is also true.

**Proposition 1.** Let $G_1$ and $G_2$ be graphs having the same order and the same size. Then $G_1 \rightarrow G_2$.

**Proof.** If $G_1 \cong G_2$, then $G_1 \rightarrow G_2$; so we may assume, without loss of generality, that $G_1 \not\cong G_2$. Suppose that $G_1$ (and $G_2$) has order $p$ and size $q$ (necessarily $p \geq 4$ and $q \geq 2$). Without loss of generality, we assume that $V(G_1) = V(G_2) = \{v_1, v_2, \ldots, v_p\}$.

For the complete graph $K_p$ having the vertex set $\{v_1, v_2, \ldots, v_p\}$, we say that an edge $v_av_b \ (a < b)$ is less than an edge $v_cv_d \ (c < d)$, written $v_av_b < v_cv_d$, if either (i) $a < c$ or (ii) $a = c$ and $b < d$. This produces a linear ordering of the edges $e_i, \ i = 1, 2, \ldots, \binom{p}{2}$, of $K_p$, where

$$(1) \quad v_1v_2 = e_1 < e_2 < e_3 < \ldots < e_{\binom{p}{2}} = v_{p-1}v_p.$$  

We say that the weight of the edge $e_i$ is $i$. Further, if $G$ is a graph having the vertex set $\{v_1, v_2, \ldots, v_p\}$, then the weight of $G$ is defined to be the sum of the weights of its edges, where the weights are determined by (1).

Define the graph $H$ to have the vertex set $\{v_1, v_2, \ldots, v_p\}$ and the $q$ smallest edges of $K_p$ as defined in (1), i.e., $E(H) = \{e_1, e_2, \ldots, e_q\}$. Note that $H$ has weight $\sum_{i=1}^{q} i$. We now show that $G_1 \rightarrow H$. Suppose, to the contrary, that $G_1$ cannot be transformed into $H$. Then let $F$ be a graph with $V(F) = \{v_1, v_2, \ldots, v_p\}$ and minimum weight $w$ such that $G_1 \rightarrow F$. Therefore, $w > \sum_{i=1}^{q} i$. This implies that there exist edges $v_av_b$ and $v_cv_d$ such that $v_av_b \notin E(F)$, $v_cv_d \in E(F)$ and $v_av_b < v_cv_d$. Let $F^* = F + v_av_b - v_cv_d$. We show that $F \rightarrow F^*$. Since $G_1 \rightarrow F$, this implies that $G_1 \rightarrow F^*$. However, since $F^*$ has smaller weight than $F$, a contradiction is produced, yielding the desired result that $G_1 \rightarrow H$. We consider two cases.
Case 1. Suppose that $a = c$. Thus $b < d$. By rotating the edge $v_cv_d$ into $v_aw_b$, the graph $F$ is transformed into $F^*$. 

Case 2. Suppose that $a < c$. If $b = d$ or $b = c$, then, as in Case 1, we may rotate the edge $v_c v_d$ into $v_b v_a$ so that $F$ is transformed into $F^*$. Assume, then, that $b \neq d$ and $b \neq c$ so that $v_a, v_b, v_c$ and $v_d$ are four distinct vertices. If $v_b v_d \notin E(F)$, then we may rotate $v_c v_d$ into $v_b v_a$, and then rotate $v_b v_d$ into $v_b v_a$, thereby concluding that $F$ can be transformed into $F^*$. If $v_b v_d \in E(F)$, then we rotate $v_b v_d$ into $v_b v_a$, after which we rotate $v_c v_d$ into $v_b v_d$, again showing that $F$ can be transformed into $F^*$.

We now have that $G_1 \rightarrow H$. Likewise, $G_2 \rightarrow H$. From this, it follows that $G_1 \rightarrow G_2$.

### DISTANCE BETWEEN GRAPHS

Let $G_1$ and $G_2$ be two graphs having the same order and the same size. We define the distance $d(G_1, G_2)$ between $G_1$ and $G_2$ as $0$ if $G_1 \cong G_2$ and, otherwise, as the smallest positive integer $n$ for which there exists a sequence $H_0, H_1, \ldots, H_n$ of graphs such that $G_1 \cong H_0$, $G_2 \cong H_n$, and $H_i$ can be transformed into $H_{i+1}$ by an edge rotation for $i = 0, 1, \ldots, n - 1$. By Proposition 1, this “distance” is a well-defined concept. Further, if $\mathcal{G}_{p,q}$ is the set of all graphs having order $p$ and size $q$, for some fixed integers $p$ and $q$, then $(\mathcal{G}_{p,q}, d)$ is a metric space.

We make the following observation concerning complements of graphs.

**Proposition 2.** Let $G_1$ and $G_2$ be two graphs having the same order and the same size. Then

$$d(G_1, G_2) = d(\bar{G}_1, \bar{G}_2).$$

**Proof.** If $d(G_1, G_2) = 0$ then $G_1 \cong G_2$, implying that $\bar{G}_1 \cong \bar{G}_2$ and $d(\bar{G}_1, \bar{G}_2) = 0$. Assume then that $d(G_1, G_2) = n \geq 1$. Hence there exists a sequence

$$G_1 \cong H_0, H_1, \ldots, H_n \cong G_2,$$

where $H_i$ can be transformed into $H_{i+1}$ by an edge rotation for $i = 0, 1, \ldots, n - 1$, where, say, $H_{i+1} = H_i - u_i v_i + u_i w_i$. Observe that $H_{i+1} = H_i - u_i w_i + u_i v_i$, i.e., $H_i$ can be transformed into $\bar{H}_{i+1}$ by an edge rotation. Thus the sequence

$$G_1 \cong \bar{H}_0, \bar{H}_1, \ldots, \bar{H}_n \cong \bar{G}_2$$

implies that $d(\bar{G}_1, \bar{G}_2) \leq d(G_1, G_2) = n$.

Now by applying the above technique to the sequence (2), we have $d(\bar{G}_1, \bar{G}_2) \leq d(\bar{G}_1, \bar{G}_2)$ or

$$n = d(G_1, G_2) \leq d(\bar{G}_1, \bar{G}_2) = n,$$

producing the desired result. 

Next we show that any nonnegative integer is the distance between some pair of graphs.
Proposition 3. For every nonnegative integer n, there exist graphs $G_1$ and $G_2$ such that $d(G_1, G_2) = n$.

Proof. If $n = 0$, then for every graph $G$, $d(G, G) = 0$, so take $G_1 = G_2 = G$. Let $n \geq 1$ be given. Let $G_1 = (n + 1)K_2$ and $G_2 = K(1, n + 1) \cup nK_1$, so that $G_1$ and $G_2$ are graphs of order $2n + 2$ and size $n + 1$. Suppose that $E(G_1) = \{u_0v_0, u_1v_1, \ldots, u_nv_n\}$. Let $H_0 = G_1$ and for $i = 0, 1, \ldots, n - 1$, define

$$H_{i+1} = H_i - u_{i+1}v_{i+1} + u_0v_{i+1}.$$ 

Note that $H_n \cong G_2$ so that $d(G_1, G_2) \leq n$. On the other hand, every edge rotation of a graph $G$ produces a graph $H$ such that $|\deg_G v - \deg_H v| \leq 1$ for every vertex $v$ of $G_1$. Since $G_1$ is 1-regular and $G_2$ contains a vertex of degree $n + 1$, at least $n$ edge rotations are required to transform $G_1$ into $G_2$. Thus $d(G_1, G_2) \geq n$ and the result follows. 

In order to present an upper bound for the distance between graphs (having the same order and size), we introduce a new concept. For nonempty graphs $G_1$ and $G_2$, we define a greatest common subgraph of $G_1$ and $G_2$ as any graph $G$ of the maximum size without isolated vertices that is a subgraph of both $G_1$ and $G_2$.

While every pair $G_1, G_2$ of nonempty graphs has a greatest common subgraph, such a subgraph need not be unique. For example, the graphs $G_1$ and $G_2$ of Figure 2 (of order 7 and size 6) have three greatest common subgraphs, namely $G$, $G'$ and $G''$. Although these subgraphs are all different, they, of course, possess the same maximum size, namely 3, in this case.

![Figure 2](image)

The main reason for introducing greatest common subgraphs lies in the following result.
Proposition 4. Let $G_1$ and $G_2$ be graphs having order $p$ and size $q \geq 1$, and let $G$ be a greatest common subgraph of $G_1$ and $G_2$, where $G$ has size $s$. Then $d(G_1, G_2) \leq 2(q - s)$.

Proof. If $s = q$, then $G_1 \cong G_2$ and $d(G_1, G_2) = 0$. Thus, we assume that $1 \leq s < q$. Let the vertices of $G_1$ and $G_2$ be labeled $v_1, v_2, ..., v_p$ so that subgraphs of $G_1$ and $G_2$ isomorphic to $G$ are identically labeled. Since $G_1 \ncong G_2$, the graph $G_1$ contains an edge $v_iv_j$ that is not in $G_2$ and $G_2$ contains an edge $v_kv_l$ that is not in $G_1$.

Suppose that $v_j = v_k$. Then $G_1$ can be transformed into $G_1' = G_1 - v_jv_j + v_kv_l$ by an edge rotation and $d(G_1, G_1') = 1$. Hence we may assume that $\{v_i, v_j\} \cap \{v_k, v_l\} = \emptyset$.

Suppose that at least one of $v_i$ and $v_j$ is not adjacent in $G_1$ to at least one of $v_k$ and $v_l$; say $v_i v_k \notin E(G_1)$. Then $G_1$ can be transformed into $G_1^* = G_1 - v_jv_j + v_kv_k$ by rotating $v_j v_j$ into $v_kv_k$, and $G_1^*$ can be transformed into $G_1^{**} = G_1^* - v_kv_k + v_kv_j$ by rotating $v_kv_k$ into $v_kv_j$. Thus $d(G_1, G_1^{**}) \leq 2$.

Assume then that each of $v_i$ and $v_j$ is adjacent to both $v_k$ and $v_l$. The graph $G_1$ can be transformed into $G_1' = G - v_jv_j + v_jv_j$ by rotating $v_jv_j$ into $v_jv_j$, and $G_1'$ can be transformed into $G_1'' = G_1' - v_jv_j + v_jv_j$ by rotating $v_jv_j$ into $v_jv_j$. Therefore, $d(G_1, G_1'') \leq 2$.

Hence, in any case, $G_1$ can be transformed into $H_1 = G_1 - v_jv_j + v_jv_j$ and $d(G_1, H_1) \leq 2$. The graphs $H_1$ and $H_2$ have $s + 1$ edges in common. Proceeding as above, we construct a graph $H_2$ such that $d(G_1, H_2) \leq 4$, and $H_2$ and $G_2$ have $s + 2$ edges in common. Continuing in this manner, we construct a graph $H_{q-s} = G_2$ such that $d(G_1, G_2) \leq 2(q - s)$.

The bound presented in the previous result cannot be improved in general, for if $n \geq 1$, define

\[ G_1 = K_{2n} \cup K_{4n^2 - 4n} \quad \text{and} \quad G_2 = (2n^2 - n)K_2. \]

Observe that each of $G_1$ and $G_2$ has order $4n^2 - 2n$ and size $q = 2n^2 - n$. In this case, $G_1$ and $G_2$ have a unique greatest common subgraph $G = nK_2$, which has size $s = n$. Therefore,

\[ 2(q - s) = 2[(2n^2 - n) - n] = 4n^2 - 4n. \]

The graph $G_2$ is 1-regular, while $G_1$ contains $4n^2 - 4n$ isolated vertices. Therefore, $d(G_1, G_2) \geq 4n^2 - 4n$. By Proposition 4, $d(G_1, G_2) \leq 2(q - s) = 4n^2 - 4n$, so that $d(G_1, G_2) = 2(q - s)$.

Reference


Authors' address: Western Michigan University, Kalamazoo, Michigan 49008, USA.