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*Časopis pro pěstování matematiky*, Vol. 110 (1985), No. 4, 337--342

Persistent URL: <http://dml.cz/dmlcz/118238>

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ON POINTS OF QUALITATIVE SEMICONTINUITY

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(Received October 5, 1983)

Let  $\mathcal{I}$  be the  $\sigma$ -ideal of sets of the first category on the real line. For a real function  $f: R \rightarrow R$  let us define the qualitative upper limit at the point  $x$

$$q\text{-lim}_{t \rightarrow x} \sup f(t) = \inf \{y \in R: \{t \in R: f(t) < y\} \text{ is residual at } x\} .$$

Similarly let us define the qualitative lower limit of  $f$  at  $x$

$$q\text{-lim}_{t \rightarrow x} \inf f(t) = \sup \{y \in R: \{t \in R: f(t) > y\} \text{ is residual at } x\} .$$

We use the notation introduced in [1]:

$$Q(f) = \{r \in R: q\text{-lim}_{t \rightarrow r} \sup f(t) = f(r) = q\text{-lim}_{t \rightarrow r} \inf f(t)\} ,$$

$$S_q(f) = \{r \in R: q\text{-lim}_{t \rightarrow r} \sup f(t) \leq f(r)\} ,$$

$$T_q(f) = \{r \in R: q\text{-lim}_{t \rightarrow r} \sup f(t) < f(r)\} ,$$

$$S_q^1(f) = \{r \in R: q\text{-lim}_{t \rightarrow r} \inf f(t) \geq f(r)\} ,$$

$$T_q^1(f) = \{r \in R: q\text{-lim}_{t \rightarrow r} \inf f(t) > f(r)\} .$$

The following facts are proved in [1].

**Fact 0.** *There exist sets  $B$  and  $C$  such that  $B$  is a  $G_\delta$  set,  $C \in \mathcal{I}$  and  $Q(f) = B - C$ .*

**Fact 1.** *The sets  $T_q(f)$  and  $T_q^1(f)$  are of the first category.*

**Fact 2.** *The sets  $S_q(f) - Q(f)$  and  $S_q^1(f) - Q(f)$  do not contain sets of the second category having the Baire property.*

Z. Grande in [1] showed Theorem 3 and stated the following Problem 1.

**Theorem 3.** Let  $A, B, C \subseteq R$  satisfy

(i)  $C \in \mathcal{S}$ ,  $B \subseteq A$ ,  $C \subseteq A - B$  and the set  $A - B$  do not contain sets of second category having the Baire property

and

(ii)  $B = D - C$ , where  $D$  is a  $G_\delta$  set. Then there exists a function  $g: R \rightarrow R$  such that  $Q(g) = B$ ,  $S_q(g) = A$  and  $T_q(g) = C$ .

**Problem 1.** Accepting the assumption (i) of Theorem 3 let us suppose furthermore that  $B = D - D_1$ , where  $D$  is a  $G_\delta$  set and  $D_1 \in \mathcal{S}$ .

Is there a function  $g: R \rightarrow R$  such that  $Q(g) = B$ ,  $S_q(g) = A$  and  $T_q(g) = C$ ?

The answer to this question is negative. It follows from the following fact.

**Fact 3.** Let  $D_q(f) = \{r \in R: q\text{-}\lim_{t \rightarrow r} \inf f(t) = q\text{-}\lim_{t \rightarrow r} \sup f(t)\}$ . Then  $D_q(f)$  is a  $G_\delta$  set for every function  $f: R \rightarrow R$ .

**Proof.** It is easy to show that for every  $a \in R$  the sets  $A = \{x \in R: q\text{-}\lim_{t \rightarrow x} \inf f(t) \leq a\}$  and  $B = \{x \in R: q\text{-}\lim_{t \rightarrow x} \sup f(t) \geq a\}$  are closed. Indeed, if  $q\text{-}\lim_{t \rightarrow x} \inf f(t) > a$  then there exist:  $\varepsilon > 0$  and a neighbourhood  $U$  of  $x$  such that  $U \cap \{y \in R: f(y) < a + \varepsilon\} \in \mathcal{S}$ . So for every  $y \in U$  we have  $q\text{-}\lim_{t \rightarrow y} \inf f(t) \geq a + \varepsilon$  and  $x \notin \text{CIA}$ . Then for all rational numbers  $p, q \in Q$  the sets  $A(f, p, q) = \{x \in R: q\text{-}\lim_{t \rightarrow x} \inf f(t) \leq p < q \leq q\text{-}\lim_{t \rightarrow x} \sup f(t)\}$  are closed. Since  $R - D_q(f) = \bigcup \{A(f, p, q): p, q \in Q\}$ ,  $D_q(f)$  is a  $G_\delta$  set.

It is clear that  $Q(f) \subseteq D_q(f)$  and  $D_q(f) - Q(f) \subseteq T_q(f) \cup T_q^1(f)$ , hence  $Q(f) = D_q(f) - [T_q(f) \cup T_q^1(f)]$ .

Assume that  $A = R$ ,  $C = \emptyset$ ,  $R - B \in \mathcal{S}$  and  $B$  is not a  $G_\delta$  set. Suppose that there exists a function  $f: R \rightarrow R$  such that  $Q(f) = B$ ,  $S_q(f) = R$  and  $T_q(f) = C$ . Then  $T_q^1(f) = \emptyset$ . If  $D$  is a  $G_\delta$  set and  $B \subseteq D$  then  $D - B$  is non empty and  $D - B \not\subseteq T_q(f) \cup T_q^1(f)$ . This is impossible since the Fact 3 holds.

In the next part we assume that every set  $A \subseteq R$  of cardinality less than continuum is of the first category. Notice that if CH (Continuum Hypothesis) or MA (Martin's Axiom) are assumed then this condition holds. [3]

The following theorem is generalization of Theorem 3 [1].

**Theorem.** (MA) For every sets  $A, A_1, B, C, C_1 \subseteq R$  the following conditions are equivalent:

- (i)  $A \cap A_1 = B$ ,  
 $C \cup C_1 \in \mathcal{S}$ ,  
 $C \subseteq A - B$ ,  $C_1 \subseteq A_1 - B$ ,

the sets  $A - B$  and  $A_1 - B$  do not contain sets of the second category having the Baire property,

there exists a  $G_\delta$  set  $D$  such that  $B = D - (C \cap C_1)$ ,

- (ii) there exists a function  $f: R \rightarrow R$  such that  $A = S_q(f)$ ,  $A_1 = S_q^1(f)$ ,  $B = Q(f)$ ,  $C = T_q(f)$  and  $C_1 = T_q^1(f)$ .

**Proof.** The implication (ii)  $\Rightarrow$  (i) follows from the facts 0 – 3.

(i)  $\Rightarrow$  (ii). Let  $E = Cl B$ . Since  $E - D \subseteq E - B$ , we have  $E - D \in \mathcal{I}$ . Notice that  $E - D$  is a  $F_\sigma$  set and  $E - D = \bigcup_{n \in N} F_n$ , where  $F_n$  are closed, nowhere dense and  $F_i \cap F_j = \emptyset$  for  $i \neq j$  [4].

Let  $(a_n)_{n \in N}$  be a sequence of positive real numbers such that  $\sum_{n \in N} a_n = 1$ .

For every  $n \in N$  we define the function  $h_n: R \rightarrow \langle -a_n, a_n \rangle$ ,

$$h_n(x) = \begin{cases} a_n \sin \frac{1}{\text{dist}(x, F_n)} & \text{for } x \notin F_n, \\ 0 & \text{for } x \in F_n. \end{cases}$$

For  $n \in N$  the function  $h_n$  is continuous on the set  $R - F_n$  and for  $x \in F_n$ ,  $q\text{-lim sup}_{t \rightarrow x} h_n(t) = a_n \geq h_n(x) \geq -a_n = q\text{-lim inf}_{t \rightarrow x} h_n(t)$ .

In the first step we define a function  $h: R \rightarrow R$  such that  $Q(h) = S_q^1(h) = S_q(h) = R - (E - D) = R - \bigcup_{n \in N} F_n$  and  $T_q(h) = T_q^1(h) = \emptyset$ . Let  $h(x) = \sum_{n \in N} h_n(x)$ . This function satisfies the above conditions.

Indeed:

a) Assume that  $x \notin \bigcup_{n \in N} F_n$ . Since  $h$  is a sum of a uniformly convergent series,  $h$  is continuous at the point  $x$ .

b) If  $x \in F_n$  then

$$\begin{aligned} q\text{-lim sup}_{t \rightarrow x} h(t) &= a_n + \sum_{m \neq n} a_m \sin \frac{1}{\text{dist}(x, F_m)} = \\ &= h(x) + a_n > h(x) > h(x) - a_n = -a_n + \sum_{m \neq n} a_m \sin \frac{1}{\text{dist}(x, F_m)} = q\text{-lim inf}_{t \rightarrow x} h(t). \end{aligned}$$

Hence  $x \notin S_q(h) \cup S_q^1(h)$ .

Assume that  $E = R$ . Then the following function  $f: R \rightarrow R$  satisfies the conditions of the theorem

$$f(x) = \begin{cases} 2 & \text{for } x \in C, \\ -2 & \text{for } x \in C_1, \\ q\text{-lim sup}_{t \rightarrow x} h(t) & \text{for } x \in A - C, \\ q\text{-lim inf}_{t \rightarrow x} h(t) & \text{for } x \in A_1 - C_1, \\ h(x) & \text{elsewhere.} \end{cases}$$

Since  $\{x \in R: f(x) \neq h(x)\} \in \mathcal{S}$ , so for every  $x \in R$ ,  $q\text{-lim sup } f(t) = q\text{-lim sup } h(t)$  and  $q\text{-lim inf } f(t) = q\text{-lim inf } h(t)$ . Hence  $B \subseteq Q(f)$ ,  $C \subseteq T_q(f)$  and  $C_1 \subseteq T_q^1(f)$ .

If  $x \in A - (B \cup C)$  then  $x \in E - D$ . Hence  $f(x) = q\text{-lim sup } h(t) = q\text{-lim sup } f(t) > q\text{-lim inf } h(t) = q\text{-lim inf } f(t)$  and  $A - (B \cup C) \subseteq S_q(f) - [Q(f) \cup T_q(f)]$ . Similarly,  $A_1 - (B \cup C_1) \subseteq S_q^1(f) - [Q(f) \cup T_q^1(f)]$  and  $R - (A \cup A_1) \subseteq R - [S_q(f) \cup S_q^1(f)]$ . Consequently,  $Q(f) = B$ ,  $S_q(f) = A$ ,  $S_q^1(f) = A_1$ ,  $T_q(f) = C$  and  $T_q^1(f) = C_1$ .

Now assume that  $R - E \neq \emptyset$ . We prove the following lemma.

**Lemma.** *If  $A$  is an open, non empty subset of  $R$  and  $B \subseteq A$  then there exists a partition  $(K_n)_{n \in N}$  of  $A$  such that sets  $K_n$  are of the second category at every point  $x \in A$  and if  $B$  is of the second category at  $x$  then  $K_n \cap B$  ( $n = 1, 2, \dots$ ) is of the second category at  $x$ .*

**Proof of lemma.** The construction of the sets  $K_n$  is very similar to the construction of Bernstein's set [2].

Let:

$(r_\xi)$  be an enumeration of the set  $A$ ,

$(I_n)_{n \in N}$  be a countable basis of  $A$ ,

$(H_{n,\eta})_{\eta < 2^{\omega_0}}$  be an enumeration of the family of the residual and  $G_\delta$  subsets of  $I_n$ ,

$$H_{n,\eta}^1 = \begin{cases} H_{n,\eta} & \text{if } H_{n,\eta} \cap B \in \mathcal{S}, \\ H_{n,\eta} \cap B & \text{if } H_{n,\eta} \cap B \notin \mathcal{S}, \end{cases}$$

$(H_\xi)_{\xi < 2^{\omega_0}}$  be an enumeration of the family  $\{H_{n,\eta}^1: n \in N, \eta < 2^{\omega_0}\}$ .

Since MA holds, so for every  $\xi$  the cardinality of  $H_\xi$  is continuum. We shall construct inductively a sequence  $(x_{\xi,n})$  of the type  $2^{\omega_0} \times \omega_0$ :

$$x_{\eta,0} = \min_{\xi} \{r_\xi: r_\xi \in H_\eta - \{x_{\gamma,k}: \gamma < \eta\}\},$$

$$x_{\eta,n} = \min_{\xi} \{r_\xi: r_\xi \in H_\eta - \{x_{\gamma,k}: (\gamma < \eta \vee (\gamma = \eta \& k < n))\}\}.$$

Let us define sets  $K_n$  as follows:

$$K_n = \begin{cases} \{x_{\eta,n}: \eta < 2^{\omega_0}\} & \text{for } n > 0, \\ A - \bigcup_{m>0} K_m & \text{for } n = 0. \end{cases}$$

It is easy to show that sets  $K_n$  ( $n = 1, 2, \dots$ ) are of the second category at every point  $x \in A$ . Suppose that the set  $B$  is of the second category at  $x$  and there exists a number  $n \in N$  such that  $K_n \cap B$  is of the first category at  $x$ . Then there exist  $I_n$  and  $H_{n,\eta}$  such that  $H_{n,\eta} \subseteq A - K_n \cap B$ . This is impossible since the set  $H_{n,\eta} \cap B \cap K_n$  is non empty.

In this step we shall construct a function  $g: R \rightarrow R$  such that  $Q(g) = B$ ,  $T_q(g) = C$ ,  $T_q^1(g) = C_1$ ,  $S_q(g) = B \cup C$  and  $S_q^1(g) = C_1 \cup B$ . Let  $(b_n)_{n \in \mathbb{N}}$  be an enumeration of the set of all rational numbers from the interval  $(-1, 1)$ .

Let  $(K_n)_{n \in \mathbb{N}}$  be a partition of  $R - E$  such that for every  $x \in R - E$  the sets  $K_n$  ( $n = 1, 2, \dots$ ) are of the second category at  $x$  and if the set  $R - (A \cup A_1)$  is of the second category at  $x$  then  $K_n - (A \cup A_1)$  is of the second category at  $x$ .

The function  $g$  is defined as follows:

$$g(x) = \begin{cases} 2 & \text{for } x \in C, \\ -2 & \text{for } x \in C_1, \\ h(x) & \text{for } x \in E - (C \cup C_1), \\ h(x) + \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)} \cdot b_n & \text{for } x \in K_n - (C \cup C_1). \end{cases}$$

a) It is clear that  $C \subseteq T_q(g)$  and  $C_1 \subseteq T_q^1(g)$ .

b) If  $x \in E - B$  then  $g(x) = h(x)$  and  $g \upharpoonright R - (C \cup C_1)$  is continuous at  $x$ . Since  $C \cup C_1 \in \mathcal{I}$ ,  $g$  is qualitative continuous at  $x$ .

c) If  $x \in (E - D) - (C \cup C_1)$  then  $g(x) = h(x)$ ,  $q\text{-lim inf}_{t \rightarrow x} g(t) = q\text{-lim inf}_{t \rightarrow x} h(t)$  and  $q\text{-lim sup}_{t \rightarrow x} h(t) = q\text{-lim sup}_{t \rightarrow x} g(t)$ . Since  $x \in R - (S_q(h) \cup S_q^1(h))$ ,  $x \in R - (S_q(g) \cup S_q^1(g))$ .

In the next step we define a function  $f: R \rightarrow R$  such that  $Q(f) = B$ ,  $S_q(f) = A$ ,  $S_q^1(f) = A_1$ ,  $T_q(f) = C$  and  $T_q^1(f) = C_1$ . Let us define the function  $f$  as follows:

$$f(x) = \begin{cases} q\text{-lim sup}_{t \rightarrow x} g(t) & \text{for } x \in A - (B \cup C), \\ q\text{-lim inf}_{t \rightarrow x} g(t) & \text{for } x \in A_1 - (B \cup C_1), \\ g(x) & \text{elsewhere.} \end{cases}$$

a) If  $x \in C$  then  $f(x) = g(x) > q\text{-lim sup}_{t \rightarrow x} g(t) \geq q\text{-lim sup}_{t \rightarrow x} f(t)$  and  $x \in T_q(f)$ .

Similarly,  $C_1 \subseteq T_q^1(f)$ .

b) Notice that for  $x \in R - E$  we have

$$\begin{aligned} q\text{-lim sup}_{t \rightarrow x} g(t) &= q\text{-lim sup}_{t \rightarrow x} h(t) + \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)} = \\ &= h(x) + \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)} \end{aligned}$$

and

$$\begin{aligned} q\text{-lim inf}_{t \rightarrow x} g(t) &= q\text{-lim inf}_{t \rightarrow x} h(t) - \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)} = \\ &= h(x) - \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)}. \end{aligned}$$

Since for  $x \in E$  the set

$$\left\{ t \in R: |f(t) - h(t)| < \frac{\text{dist}(t, E)}{1 + \text{dist}(t, E)} \right\}$$

is residual at the point  $x$ , we have  $q\text{-lim sup } f(t) = q\text{-lim sup } h(t)$  and  $q\text{-lim inf } f(t) = q\text{-lim inf } h(t)$ . Hence  $C_q(f) \cap [E - (C \cup C_1)] = B$ ,  $S_q(f) \cap [E - (C \cup C_1)] = A - C$  and  $S'_q(f) \cap [E - (C \cup C_1)] = A_1 - C_1$ .

c) Assume that  $x \in R - (E \cup C \cup C_1)$ . The following cases may occur: The set  $R - (A \cup A_1)$  is of the second category at  $x$ . Then for every  $n \in N$  the set  $K_n - (A \cup A_1)$  is of the second category at  $x$ ,

$$q\text{-lim sup}_{t \rightarrow x} f(t) = h(x) + \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)} \quad \text{and} \quad q\text{-lim inf}_{t \rightarrow x} f(t) = h(x) - \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)}.$$

There exists a neighbourhood  $U \subseteq R - E$  of  $x$  such that  $U - (A \cup A_1) \in \mathcal{I}$ . Since the sets  $A - B$  and  $A_1 - B$  do not contain sets of the second category having the Baire property, hence the sets  $A - B$  and  $A_1 - B$  are of the second category at  $x$ . Then

$$q\text{-lim sup}_{t \rightarrow x} f(t) = \lim_{t \rightarrow x} \left( h(t) + \frac{\text{dist}(t, E)}{1 + \text{dist}(t, E)} \right) = h(x) + \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)}$$

and

$$q\text{-lim inf}_{t \rightarrow x} f(t) = h(x) - \frac{\text{dist}(x, E)}{1 + \text{dist}(x, E)}.$$

Thus, if  $x \in A - C$  then  $x \in S_q(f) - [Q(f) \cup T_q(f)]$ , if  $x \in A_1 - C_1$  then  $x \in S'_q(f) - [Q(f) \cup T'_q(f)]$  and if  $x \notin A \cup A_1$  then  $x \notin S_q(f) \cup S'_q(f)$ .

Therefore  $f$  satisfies the condition (ii).

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