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Časopis pro pěstování matematiky, Vol. 110 (1985), No. 4, 394--402

Persistent URL: http://dml.cz/dmlcz/118239

Časopis pro pěstování matematiky, roč. 110 (1985), Praha

CONTINUOUS DEPENDENCE FOR SEMILINEAR PARABOLIC FUNCTIONAL EQUATIONS WITHOUT UNIQUENESS

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(Received April 16, 1984)

1. INTRODUCTION

In [4] we have proved the existence theorem for an abstract semilinear equation with infinite delay

$$\dot{u}(t) + A\,u(t) = f(t, u_t) + \int_{t_0}^{t} g(t, s, u(s))\,\mathrm{d}s\,, \tag{1.1}$$

$$u_{t_0} = \varphi\,. \tag{1.2}$$

Henceforth u_t denotes the shift of a function u, i.e. $u_t(s) = u(t + s)$, $s \in (-\infty, 0]$. The assumptions of the theorem do not guarantee the uniqueness of a solution to the problem (1.1), (1.2), thus the question of continuous dependence is more delicate than in the cases of uniqueness. Recently, theorems of this type for the equation (1.1) have been proved in [2], [5].

In the present paper, the continuous dependence theorem (Theorem 2) is proved in case of nonuniquess of a solution. As an important step of the proof a Kneser type theorem on compactness of the funnel of solutions (Theorem 1) is established.

The operator A will be supposed to satisfy

(H 1) *A is a sectorial operator in a Banach space X,*

(H 2) *A^{-1} is a compact operator in X,*

$$\inf\{\operatorname{Re}\lambda;\ \lambda \in \sigma(A)\} > 0\,. \tag{1.3}$$

If A is sectorial then $-A$ generates an analytic C_0-semigroup which will be denoted by e^{-At}, $t \geqq 0$. Further, all real powers A^α are defined and if X^α denote their domains endowed with the norms $\|x\|_\alpha = \|A^\alpha x\|_X$, we obtain Banach spaces and the following estimates of the semigroup in these spaces (see e.g. [1], [3]):

$$(\forall \alpha \geqq 0\ \exists K_1(\alpha)\ \forall x \in X\ \forall t > 0) \Rightarrow \|\mathrm{e}^{-At}x\|_\alpha \leqq K_1(\alpha)\, t^{-\alpha}\|x\|\,, \tag{1.4}$$

$$(\forall \alpha \in [0, 1]\ \exists K_2(\alpha)\ \forall x \in X^\alpha\ \forall t > 0) \Rightarrow \tag{1.5}$$

$$\Rightarrow \|(\mathrm{e}^{-At} - I)\,x\| \leqq K_2(\alpha)\, t^\alpha \|x\|_\alpha\,.$$

Let us note that if the sectorial operator A does not satisfy the assumption (1.3), then the spaces X^α are defined in terms of the powers of the operator $A + aI$, which satisfies (1.3). The estimates (1.4), (1.5) for the semigroup e^{-At} hold on compact intervals, which is sufficient for our purposes. The condition (1.3) is introduced solely for simplification of the exposition. Let us repeat some properties of the operators satisfying (H 1), (H 2).

Lemma 1. *LetA satisfy* (H 1). *Then the following statements are equivalent:*

(i) *A satisfies* (H 2).

(ii) e^{-At} *is a compact operator in X for any $t > 0$.*

(iii) *For $0 \leqq \beta < \alpha$ the natural embeding $X^\alpha \to X^\beta$ is compact.*

The following assumptions concern nonlinear operators f, g. For $T \geqq 0$ denote by $Y^\alpha(T)$ the Banach space of all bounded, uniformly continuous maps of the interval $(-\infty, T]$ into X^α with the norm $\|u\|_{Y^\alpha(T)} = \sup_{t\in(-\infty,T]} \|u(t)\|_\alpha$.

(H 3) *There exist an open subset $U_1 \subset [t_0, +\infty) \times Y^1(0)$ and $\beta > 0$ such that f is continuous map of U_1 into X^β.*

(H 4) *There exists an open subset $U_2 \subset \{[t, s]; t_0 \leqq s \leqq t < +\infty\} \times X^1$ such that*

(i) *g is continuous map of U_2 into X,*

(ii) *g is locally Hölder continuous in the first variable and locally Lipschitz continuous in the third one on U_2.*

The initial value is required to satisfy

(H 5) $(t_0, \varphi) \in U_1$, $(t_0, t_0, \varphi(0)) \in U_2$.

The local existence of a solution is proved in [4] under the assumptions (H 1)–(H 5). This solution is a mild solution as well, i.e. it satisfies the integral equation

$$u(t) = e^{-A(t-t_0)}\varphi(0) + \int_{t_0}^{t} e^{-A(t-s)}\left[f(s, u_s) + \int_{t_0}^{s} g(s, \sigma, u(\sigma))\, d\sigma\right] ds \tag{1.6}$$

with the initial condition (1.2), for t in the interval of existence.

Note that the existence theorem for a mild solution can be proved even by replacing X^1, $Y^1(0)$ respectively by X^α, $Y^\alpha(0)$, $\alpha < 1$ and taking $\beta = 0$. However, the case $\alpha = 1$ (for example, g depends on Δu if $A = -\Delta$) is the most interesting. Further, for continuous, X^1 – valued solution to the problem (1.6), (1.2), the assumptions (H 1)–(H 4) guarantee regularity, i.e. such solutions satisfy the equation (1.1). Thus, in the sequel, we shall deal with the integral equation (1.6).

A solution $u: [t_0, \tau) \to X^1$ can be continued if $\lim u(t) = u(\tau)$ exists for $t \nearrow \tau$ and $(\tau, u_\tau) \in U_1$, $(\tau, \tau, u(\tau)) \in U_2$. In fact, if v is a solution of

$$v(t) = e^{-A(t-\tau)}u(\tau) + \int_{\tau}^{t} e^{-A(t-s)}\Big[f(s, v_s) + \tag{1.7}$$

$$+\int_{\tau}^{s} g(s, \sigma, v(\sigma))\,\mathrm{d}\sigma + \int_{t_0}^{\tau} g(s, \sigma, v(\sigma))\,\mathrm{d}\sigma\Big]\,\mathrm{d}s\,,$$

$$v_\tau = u_\tau \tag{1.8}$$

on $[\tau, t_1)$ then the joint of u and v is a solution to the problem (1.1), (1.2) on $[t_0, t_1)$. So there is a maximal, noncontinuable solution of the problem.

2. PRELIMINARIES

For $Q = (Q_1, Q_2)$, where $Q_1 \subset U_1$, $Q_2 \subset U_2$ denote $\mathscr{F}(Q) = \{(f, g); f$ satisfies (H 3), g satisfies (H 4), there exist positive constants M, L, γ such that

$$\|f(t, \psi)\|_\beta \leqq M \quad \text{for any} \quad (t, \psi) \in Q_1\,,$$

$$\|g(t_1, s, x_1)\| \leqq M \quad \text{and} \quad \|g(t_1, s, x_1) - g(t_2, s, x_2)\| \leqq \tag{2.1}$$

$$\leqq L(|t_1 - t_2|^\gamma + \|x_1 - x_2\|_1) \quad \text{for all} \quad [t_i, s, x_i] \in Q_2\}\,,$$

$\mathscr{F}_{M,L,\gamma}(Q) = \{(f, g) \in \mathscr{F}(Q); f, g$ satisfy (2.1)$\}$.

For $T > 0$ denote by $Z(T, Q)$ the set of all functions $u \in Y^1(t)$ such that $(t, u_t) \in Q_1$, $(t, s, u(s)) \in Q_2$ for all $t_0 \leqq s \leqq t < T$. Let f, g satisfy (H 3), (H 4). Then for $u \in Z(T, Q)$, $t_0 \leqq t < T$, let us define the following operators:

$$G(t, u) = \int_{t_0}^{t} g(t, s, u(s))\,\mathrm{d}s\,, \tag{2.2}$$

$$\Phi_1(t, u) = \int_{t_0}^{t} A\mathrm{e}^{-A(t-s)} f(s, u_s)\,\mathrm{d}s\,,$$

$$\Phi_2(t, u) = \int_{t_0}^{t} A\mathrm{e}^{-A(t-s)} G(s, u)\,\mathrm{d}s = \tag{2.3}$$

$$= \int_{t_0}^{t} A\mathrm{e}^{-A(t-s)}[G(s, u) - G(t, u)]\,\mathrm{d}s + [I - \mathrm{e}^{-A(t-t_0)}]\, G(t, u)\,.$$

We denote the operators on the right hand side of the last equality by $H_1(t, u)$ and $H_2(t, u)$. For $(f, g) \in \mathscr{F}_{M,L,\gamma}(Q)$, $u, v \in Z(T, Q)$ we have

$$\|G(t_1, u) - G(t_2, v)\| \leqq L(t_2 - t_0)\,[(t_2 - t_1)^\gamma + \tag{2.4}$$

$$+ \sup_{s\in[t_0,t_2]} \|u(s) - v(s)\|_1 + M(t_2 - t_1)$$

for $t_0 \leqq t_1 \leqq t_2 < T$, and

$$\|H_2(t, u) - H_2(t, v)\| \leqq K_2(0)\, L(t - t_0)\, \|u - v\|_{Y^1(t)} \tag{2.5}$$

for $t_0 \leqq t < T$.

Lemma 2. *Let* $(f, g) \in \mathscr{F}_{M,L,\gamma}(Q)$, $u \in Z(t, Q)$, $t_0 < t < +\infty$ *and* $\delta < \min(\beta, \gamma)$. *Then there is a function c increasing in all its arguments and such that the estimate*

$$\|\Phi_i(t_1, u) - \Phi_i(t_2, u)\| \leqq c(M, L, t)(t_2 - t_1)^\delta \tag{2.6}$$

holds for any $t_0 \leqq t_1 \leqq t_2 < t$ *and* $i = 1, 2$.

Proof. According to (1.4) and (1.5) we have

$$\|\Phi_1(t_2, u) - \Phi_1(t_1, u)\| \leqq$$

$$\leqq \int_{t_0}^{t_1} \|[e^{-A(t_2-t_1)} - I] e^{-A(t_1-s)} f(s, u_s)\|_1 \, ds + \int_{t_1}^{t_2} \|e^{-A(t_2-s)} f(s, u_s)\|_1 \, ds \leqq$$

$$\leqq K_2(\delta) K_1(1 + \delta - \beta) \frac{(t - t_0)^{\beta-\delta}}{\beta - \delta} M(t_2 - t_1)^\delta +$$

$$+ \frac{1}{\beta} K_1(1 - \beta) M(t - t_0)^{\beta-\delta} (t_2 - t_1)^\delta .$$

Moreover, using (2.3) and (2.4) we get

$$\|\Phi_2(t_2, u) - \Phi_2(t_1, u)\| \leqq$$

$$\leqq \int_{t_0}^{t_1} \|A[e^{-A(t_2-t_1)} - I] e^{-A(t_1-s)} [G(s, u) - G(t_1, u)]\| \, ds +$$

$$+ \int_{t_1}^{t_2} \|A e^{-A(t_2-s)} [G(s, u) - G(t_2, u)]\| \, ds +$$

$$+ \|[I - e^{-A(t_2-t_1)}] e^{-A(t_1-t_0)} G(t_1, u)\| +$$

$$+ \|[I - e^{-A(t_2-t_1)}] [G(t_2, u) - G(t_1, u)]\| \leqq$$

$$\leqq K_2(\delta) K_1(1 + \delta) \left(\frac{L}{\gamma - \delta} (t - t_0)^{1-\delta+\gamma} + \frac{M}{1 - \delta} (t - t_0)^{1-\delta}\right)\Big] (t_2 - t_1)^\delta +$$

$$+ K_1(1) \left[\frac{L}{\gamma} (t - t_0)^{1+\gamma-\delta} + M(t - t_0)^{1-\delta}\right] (t_2 - t_1)^\delta +$$

$$+ K_2(\delta) K_1(\delta) M(t - t_0)^{1-\delta} (t_2 - t_1)^\delta +$$

$$+ K_2(0) [L(t - t_0)^{1+\gamma-\delta} + M(t - t_0)^{1-\delta}] (t_2 - t_1)^\delta .$$

Corollary 1. *Let* $u \in Z(T, Q)$ *be a solution to the equation* (1.1) *and* $(f, g) \in \mathscr{F}(Q)$. *Then for* $t \nearrow T$ *the limits* $\lim u(t)$ *in the space* X^1 *nad* $\lim u_t$ *in the space* $Y^1(0)$ *exist.*

Proof. The function u satisfies also the integral equation (1.6). As e^{-At} is a C_0-semigroup, the limit $\lim e^{-A(t-t_0)} A\, \varphi(0)$ exists. According to Lemma 2, the limits $\lim \Phi_1(t, u)$, $\lim \Phi_2(t, u)$ exist as well and this implies the existence of $\lim u(t)$. If this limit is denoted by $u(T)$, one can easily see that $\lim u_t = u_T$.

Corollary 2. *Let $(f, g) \in \mathscr{F}(Q)$ whenever $Q_1 \subset U_1$, $Q_2 \subset U_2$ are bounded and closed. Let u be a maximal solution to the problem* (1.1), (1.2) *with the domain of definition $[t_0, \tau)$, where $\tau < +\infty$. Then there is a sequence $t_n \nearrow \tau$ such that either $(t_n, u_{t_n}) \to \partial U_1$ or $(t_n, t_n, u(t_n)) \to \partial U_2$.*

Proof. Choose sequences $Q_1^n \subset U_1$, $Q_2^n \subset U_2$ of closed bounded sets such that $U_1 = \bigcup_n Q_1^n$, $U_2 = \bigcup_n Q_2^n$, let $\tau_n \nearrow \tau$. Set $R_1^n = Q_1^n \cup \{(t, u_t);\ t \in [t_0, \tau_n]\}$, $R_2^n = Q_2^n \cup \{(t, s, u(s));\ t_0 \leqq s \leqq t \leqq \tau_n\}$, $R^n = (R_1^n, R_2^n)$. If $u \in Z(\tau, R^n)$ for some n, then according to Corollary 1, $\lim_{t \nearrow \tau} u(t) = u(\tau)$ and $\lim_{t \nearrow \tau} u_t = u_\tau$ exist. As Q_1^n, Q_2^n are closed we get $(\tau, u_\tau) \in Q_1^n$, $(\tau, \tau, u(\tau)) \in Q_2^n$, which contradicts the maximality of u.

Corollary 3. *Let the assumptions of Corollary 2 be fulfilled with $U_1 = [t_0, +\infty) \times Y^1(0)$, $U_2 = \{(t, s) \in \mathbb{R}^2;\ t_0 \leqq s \leqq t < +\infty\} \times X^1$ and let u be a maximal solution of* (1.1) *defined on $[t_0, \tau)$ with $\tau < +\infty$. Then there is a sequence t_n, $t_n \nearrow \tau$ such that $\|u(t_n)\|_1 \to +\infty$.*

Proof. It suffices to take $Q_1^n = [t_0, t_0 + n] \times \{\psi \in Y^1(0);\ \|\psi\|_{Y^1(0)} \leqq n\}$ and Q_2^n analogously as in the proof of Corollary 2.

The next lemma concerns a simultaneous continuation of all solutions starting in a neighbourhood of a compact set of initial values.

Lemma 3. *Let $Q_i \subset U_i$, $i = 1, 2$, be open sets, $\Phi \subset Y^1(0)$ a compact set such that $(t, \varphi) \in Q_1$, $(t, s, \varphi(0)) \in Q_2$ for any $t_0 \leqq s \leqq t \leqq b$, $\varphi \in \Phi$, and let M, L, γ be positive constants. Then there exist $\Delta > 0$, $\varepsilon > 0$ such that the following continuation property is valid: Whenever $(f, g) \in \mathscr{F}_{M,L,\gamma}(Q)$ and v is a maximal solution to* (1.1) *for which there is $\tau \in [t_0, b]$ such that $v \in Z(\tau, Q)$, $\operatorname{dist}(v_\tau, \Phi) < \varepsilon$, then v is defined at least on the interval $[t_0, \tau + \Delta)$ and $v \in Z(\tau + \Delta, Q)$.*

Proof. As Q_1, Q_2 are open sets and Φ is compact, there are $\nu > 0$, $T > b$ such that

$$Q_1(\nu) = \{(t, \psi);\ t_0 \leqq t \leqq T,\ \operatorname{dist}(\psi, \Phi) \leqq \nu\} \subset Q_1\,,$$
$$Q_2(\nu) = \{(t, s, x);\ t_0 \leqq s \leqq t \leqq T,\ \inf_{\varphi \in \Phi} \|x - \varphi(0)\|_1 \leqq \nu\} \subset Q_2\,.$$

Let v be a solution to (1.1) and let $\varphi \in \Phi$ be such that $\|v_\tau - \varphi\|_{Y^1(0)} < \varepsilon$. Denote by ω_ψ the module of continuity of the function ψ. According to the above, results, v can be continued whenever $(t, v_t) \in Q_1$, $(t, t, v(t)) \in Q_2$. For the distance $\operatorname{dist}(v_t, \Phi)$, $t \geqq \tau$ we get the following estimate:

$$\|v_t - \varphi\|_{Y^1(0)} = \sup_{s \in (-\infty, 0]} \|v(t + s) - \varphi(s)\|_1 \leqq$$
$$\leqq \max\{\sup_{s \in (-\infty, \tau - t]} (\|v_\tau(t + s - \tau) - \varphi(t + s - \tau)\|_1 + \|\varphi(t + s - \tau) - \varphi(s)\|_1)\,,$$
$$\sup_{s \in [\tau - t, 0]} (\|v(t + s) - v(\tau)\|_1 + \|v(\tau) - \varphi(0)\|_1 + \|\varphi(0) - \varphi(s)\|_1)\} \leqq$$

$$\leqq \|v_\tau - \varphi\|_{Y^1(0)} + \omega_\varphi(t - \tau) + K_1(0) \sup_{h\in[0,t-\tau]} \|(e^{-Ah} - I)\,\varphi(0)\|_1 +$$

$$+ \omega_{\Phi_1(.,v)}(t - \tau) + \omega_{\Phi_2(.,v)}(t - \tau)\,.$$

By the compactness of Φ, the estimates (2.6) of Lemma 2 and the properties of the C_0-semigroup e^{-At}, we can find Δ and ε such that $\varepsilon < v/5$ and

$$\max\{\sup_{\varphi\in\Phi} \omega_\varphi(\Delta),\ c(M, L, T)\,\Delta^\delta,\ K_1(0) \sup_{h\in[0,\Delta]} \|(e^{-Ah} - I)\,A\,\varphi(0)\|\} < \frac{v}{5}\,.$$

Therefore $\|v_t - \varphi\|_{Y^1(0)} < v$ and the assertion of Lemma 3 follows.

3. COMPACTNESS OF THE FUNNEL

In this section we shall investigate the equation (1.1) with the fixed initial condition (1.2). We denote by $\mathfrak{M}$ the set of all maximal solutions to the problem (1.1), (1.2) and by $F(a)$ the funnel of solutions up to the point a, i.e.

$$F(a) = \{(t, u(t));\ t \in [t_0, a],\ u \in \mathfrak{M}\}\,.$$

Lemma 4. *Let $(f, g) \in \mathscr{F}(Q)$ whenever Q is bounded and let all solutions of (1.1), (1.2) exist on the interval $[t_0, b]$. Let $F(b)$ be bounded in the space $\mathbb{R} \times X^1$. Then $\mathfrak{M}$ is compact in $C([t_0, b], X^1)$.*

Proof. We proceed according to the generalized Arzela-Ascoli theorem, i.e., we shall prove:

(i) $\mathfrak{M}$ is closed in $C([t_0, b], X^1)$;

(ii) the elements of $\mathfrak{M}$ are uniormly equicontinuous on $[t_0, b]$;

(iii) the cross-section $\{u(t);\ u \in \mathfrak{M}\}$ is relatively compact in X^1 for each $t \in [t_0, b]$.

It is easily seen that (i) holds, (ii) follows immediately from Lemma 2. It remains to prove (iii). By virtue of (1.6), (2.2), (2.3) we can write

$$A\,u(t) - H_2(t, u) = e^{-A(t-t_0)} A\,\varphi(0) + \Phi_1(t, u) + H_1(t, u)\,.$$

The sets $\{\Phi_1(t, u);\ u \in \mathfrak{M}\}$, $\{H_1(t, u);\ u \in \mathfrak{M}\}$ are bounded in X^δ for $0 < \delta < \min(\beta, \gamma)$, $t_0 \leqq t \leqq b$, and consequently they are compact in X. The estimate (2.5) shows that $H_2(t, \cdot)$ is Lipschitz continuous with a constant less than one for $t - t_0 \leqq \Delta_1 < (K_2(0)\,L)^{-1}$, and $H_3(t, u) = A\,u(t) - H_2(t, u)$ has the property $\|H_3(t, u) - H_3(t, v)\| \geqq (1 - \Delta_1 K_2(0)\,L)\,\|u - v\|_{Y^1(t)}$. Hence each sequence $u_n(t)$, where $u_n \in \mathfrak{M}$, $t_0 \leqq t \leqq t_0 + \Delta_1$, contains a convergent subsequence and, consequently $\mathfrak{M}$ is compact in $C([t_0, t_0 + \Delta_1], X^1)$. If $t_1 \leqq t \leqq t_1 + \Delta_1$, $t_1 = t_0 + \Delta_1$, we write $H_2(t, u)$ in the form

$$H_2(t, u) = (I - e^{-A(t-t_0)}) \left[\int_{t_0}^{t_1} g(t, s, u(s))\,ds + \int_{t_1}^{t} g(t, s, u(s))\,ds\right].$$

The second term is again a contractive map and the set

$$\left\{(I - e^{-A(t-t_0)}) \int_{t_0}^{t_1} g(t, s, u(s))\, ds ; \quad u \in \mathfrak{M}\right\}$$

is compact, as has been just proved. The same argument as above yields after a finite number of steps that $\{u(t); u \in \mathfrak{M}\}$ is compact for all $t \in [t_0, b]$. This completes the proof of the lemma.

Theorem 1. *Let the assumptions* (H 1)–(H 5) *be fulfilled with* $U_1 = [t_0, +\infty) \times Y^1(0)$, $U_2 = \{(t, s) \in \mathbb{R}^2;\ t_0 \leqq s \leqq t < +\infty\} \times X^1$. *Let* $(f, g) \in \mathcal{F}(Q)$ *for each bounded* $Q = (Q_1, Q_2)$, $Q \subset (U_1, U_2)$. *Let* $b > t_0$ *be such that all solutions to the problem* (1.1), (1.2) *are defined at least on* $[t_0, b]$. *Then* $\mathfrak{M}$ *is compact in* $C^1([t_0, b], X^1)$.

Proof. In accordance with Lemma 4 we are to prove the boundedness of $F(b)$ in $\mathbb{R} \times X^1$. Suppose $t^* = \inf\{t; F(t) \text{ is unbounded}\}$. It follows from Lemma 3 that $t^* > t_0$. If $t^* \leqq b$ and $F(t^*)$ were bounded, then $\mathfrak{M}$ would be compact in $C^1([t_0, t^*], X^1)$ and thus $\{u_{t^*}; u \in \mathfrak{M}\}$ would be compact in $Y^1(0)$, and, according to Lemma 3, $F(t^* + \Delta)$ would be bounded for some $\Delta > 0$, which contradicts the definition of t^*. Hence we can find a sequence $t_n \nearrow t^*$ and $u_n \in \mathfrak{M}$ such that $\|u_n'(t_n)\|_1 \to \infty$. As $F(t)$ is bounded and consequently $\mathfrak{M}$ is compact in $C([t_0, t], X^1)$ for all $t < t^*$, there is a subsequence, denoted again by u_n, which converges to a solution $u \in \mathfrak{M}$ uniformly on each interval $[t_0, t]$ where $t < t^*$. Now we can set

$$\Phi = \{u_t;\ t \in [t_0, t^*]\},$$

$$Q_1 = [t_0, b + 1) \times \{\psi \in Y^1(0);\ \inf_{t \in [t_0, t^*]} \|\psi - u_t\|_{Y^1(0)} < 1\},$$

$$Q_2 = \{(t, s, x);\ 0 \leqq s \leqq t < b + 1,\ \inf_{\tau \in [t_0, t^*]} \|x - u(\tau)\|_1 < 1\}$$

and use Lemma 3. Let ε, Δ be the corresponding values given by Lemma 3. By the uniform convergence of the sequence u_n on the interval $[t_0, t^* - \Delta]$, the functions u_n belong eventually to $Z(t^* - \Delta, Q)$ and $\operatorname{dist}((u_n)_{t^*-\Delta}, \Phi) < \varepsilon$. It follows that $u_n \in Z(t^*, Q)$ eventually, which is a contradiction.

Remark 1. The compactness of $\mathfrak{M}$ in $C([t_0, b], X^1)$ implies the compactness of the set $\{(t, u_t);\ t \in [t_0, b],\ u \in \mathfrak{M}\}$ in $\mathbb{R} \times Y^1(0)$ and of $\{(t, s, u(s));\ t_0 \leqq s \leqq t \leqq b,\ u \in \mathfrak{M}\}$ in $\mathbb{R}^2 \times X^1$.

Corollary 4. *Let the assumptions of Theorem* 1 *be satisfied and let all maximal solutions of the problem* (1.1), (1.2) *be defined on* $[t_0, b]$. *Then there is* $\Delta > 0$ *such that they are defined on* $[t_0, b + \Delta)$.

Corollary 5. *Let the assumptions of Theorem* 1 *be satisfied and*

(3.1) $$T = \sup\{b;\ F(b) \text{ is bounded in } \mathbb{R} \times X^1\}.$$

Then either $T = +\infty$ and all maximal solutions are defined on $[t_0, +\infty)$ or $T < +\infty$ and all maximal solutions are defined at least on the interval $[t_0, T)$ and there is a solution which is not defined on any larger interval.

Proof. If $u \in \mathfrak{M}$ were defined on $[t_0, \tau)$, $\tau < T$, then Corollary 3 would give a contradiction to the boundedness of $F(\tau)$. The rest of the proof repeats the arguments of the proof of Theorem 1.

4. CONTINUOUS DEPENDENCE

In this section we shall deal with the sequence of equations of the type (1.1)

$$\dot{u}(t) + A\, u(t) = f_n(t, u_t) + \int_{t_0}^{t} g_n(t, s, u(s))\, \mathrm{d}s\,, \tag{4.1n}$$

$$u_{t_0} = \varphi_n\,. \tag{4.2n}$$

Theorem 2. *Let* (H 1)–(H 5) *be satisfied for A, f_n, g_n, φ_n, $n = 1, 2, \ldots$ with $U_1 = [t_0, +\infty) \times Y^1(0)$, $U_2 = \{(t, s) \in \mathbb{R}^2;\ t_0 \leqq s \leqq t < +\infty\} \times X^1$. Let f, g satisfy the assumptions of Theorem* 1 *and let the following conditions be fulfilled*:
(i) $\varphi_n \to \varphi$ *in* $Y^1(0)$;
(ii) *for each $(t, \psi) \in U_1$, $(t, s, x) \in U_2$ there exist neighbourhoods V_1, V_2 such that $(f_n, g_n) \in \mathscr{F}((V_1, V_2))$ eventually*;
(iii) *if $t \in [t_0, +\infty)$, $\psi_n \to \psi$ in $Y^1(0)$, then $f_n(t, \psi_n) \to f(t, \psi)$ in X^β;*
if $(t, s) \in \mathbb{R}^2$, $t_0 \leqq s \leqq t$, $x_n \to x$ in X^1, then $g_n(t, s, x_n) \to g(t, s, x)$ in X.
Let v_n be a maximal solution to the problem (4.1n), (4.2n) *and let T be given by* (3.1). *Then for each $b \in [t_0, T)$ there exists a subsequence v_{n_k} which converges to a solution of the problem* (1.1), (1.2) *in the space $C([t_0, b], X^1)$.*

Proof. According to Theorem 1 and Remark 1, the sets

$$Q_1 = \{(t, u_t);\ t \in [t_0, b],\ u \in \mathfrak{M}\}\,,$$
$$Q_2 = \{(t, s, u(s));\ t_0 \leqq s \leqq t \leqq b,\ u \in \mathfrak{M}\}$$

are compact. The assumption (ii) yields the existence of neighbourhoods Q_i^δ of Q_i, $i = 1, 2$, and constants M, L, γ, n_0 such that $(f_n, g_n), (f, g) \in \mathscr{F}_{M,L,\gamma}(Q^\delta)$ for $n \geqq n_0$. Now we set $\Phi = \{u_t;\ t \in [t_0, b],\ u \in \mathfrak{M}\}$ in Lemma 3. First, we set $\tau = t_0$ and get the existence interval $[t_0, t_0 + \Delta)$ for all v_n with n sufficiently large. In a similar way as in Lemma 4 we can prove compactness of the sequence v_n on the interval $[t_0, t_1]$, $t_1 < t_0 + \Delta$, because $(t, (v_n)_t) \in Q_1^\delta$ and the operators $H_2(t, u)(g_n)$ are contractive uniformly with respect to n. We take a subsequence of v_n converging uniformly on $[t_0, t_1]$. The dominance convergence theorem together with (ii), (iii) guarantees that the limit is a solution of (1.1), (1.2). Now, we can apply Lemma 3 with $\tau = t_1$ $n \geqq n_1 \geqq n_0$ and after a finite number of steps the assertion of Theorem 2 follows.

Remark. If the assumptions of Theorem 2 are satisfied and the problem (1.1) (1.2) possesses a unique solution, then every sequence v_n converges uniformly to this solution in the norm of X^1 on any compact subinterval of $[t_0, T)$.

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