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NOTE ON QUASI HAMILTONIAN SEMIGROUPS

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Following A. Cherubini and A. Varisco [1] a semigroup is said to be *quasi hamiltonian* if all its subsemigroups are permutable. In this note we shall show that every variety of quasi hamiltonian semigroups is commutative.

Let a be an element of a semigroup S. By $[a]$ we denote the subsemigroup of S generated by a. It is easy to show (see Lemma 3 of [2]) that a semigroup S is quasi hamiltonian if and only if we have

(1) $$ab \in [b]\,[a] \quad \text{for every} \quad a, b \in S\,.$$

A semigroup S is called *quasicommutative* [3] if we have

(2) $$ab \in [b]\, a \quad \text{for every} \quad a, b \in S\,.$$

A semigroup S is *σ-reflexive* if for every $a, b \in S$ and every subsemigroup H of S, $ba \in H$ implies $ab \in H$. It is easy to prove that a semigroup S is σ-reflexive if and only if we have

(3) $$ab \in [ba] \quad \text{for every} \quad a, b \in S\,.$$

In [4] it has been shown that the class of all quasicommutative semigroups coincides with the class of all σ-reflexive semigroups. This together with (2) and (3) implies that a semigroup S is quasicommutative if and only if we have

(4) $$ab \in b[a] \quad \text{for every} \quad a, b \in S\,.$$

Theorem 1. *Let S be a noncommutative semigroup such that $S \times S$ is a quasi hamiltonian semigroup. Then S is a periodic semigroup.*

Proof. Let $S \times S$ be a quasi hamiltonian semigroup. Suppose that $ab \neq ba$ for some $a, b \in S$. By way of contradiction, assume that there exists a non periodic element c of S. According to (1), we have $(a, c)(b, c) = (b, c)^m (a, c)^n$ for some positive integers m, n. Then we obtain $m = 1 = n$. This implies that $ab = ba$, which is a contradiction. Hence S is a periodic semigroup.

Lemma. *Let S be a quasi hamiltonian semigroup, $a, b, e \in S$ and $a^k = e = e^2$ for a positive integer k. If in $S \times S$ we have $(a, a)(b, e) = (b, e)^m (a, a)^n$ for some positive integers m, n, then $(a, a)(b, e) = (b, e)^m (a, a)$.*

Proof. Suppose that $(a, a)(b, e) = (b, e)^m (a, a)^n$. We have

$$ab = b^m a^n \quad \text{and} \quad ae = ea^n . \tag{5}$$

We can suppose that $n > 1$. Since S is quasi hamiltonian, by (1) we have $ba = a^i b^j$ for some positive integers i, j and so $ba \in S^1 abS^1$, where S^1 denotes the semigroup S with an identity adjoined. According to (1), we obtain

$$ba \in S^1 ab . \tag{6}$$

By induction we shall prove the following proposition:

$$ab \in S^1 ba(a^{n-1})^r \quad \text{for all positive integers } r . \tag{7}$$

Evidently (7) is true for $r = 1$ by (5). Suppose that (7) is true for some r. Then by (6) we have $ab \in S^1 ab(a^{n-1})^r$ and so by (5) we obtain $ab \in S^1 b^m a^n (a^{n-1})^r \subseteq \subseteq S^1 ba(a^{n-1})^{r+1}$.

It is clear that $r(n - 1) > k$ for a positive integer r. Then, by (7), we have $ab \in Se$ and so, by (6), we have $ba \in S^1 ab \subseteq Se$. Therefore $ba = bae$ and using (5) we get $ab = abe = b^m a^n e = b^m ea^n = b^m ae = b^m a$. Hence we have $(a, a)(b, e) = = (b, e)^m (a, a)$.

Theorem 2. *Let S be a semigroup such that $S \times S$ is a quasi hamiltonian semigroup. Then S is a quasicommutative semigroup.*

Proof. Suppose that $S \times S$ is a quasi hamiltonian semigroup. It is easy to show that S is a quasi hamiltonian semigroup. We can assume that S is non commutative. It follows from Theorem 1 that S is periodic. Let $a, b \in S$. Then there exists a positive integer k such that $a^k = e = e^2$. According to (1), we have $(a, a)(b, e) = (b, e)^m . . (a, a)^n$ for some positive integers m, n. Using Lemma we get $(a, a)(b, e) = = (b, e)^m (a, a)$ and so $ab = b^m a$. Then, by (2), S is quasicommutative.

Theorem 3. *Let S be a semigroup such that $S \times S$ is a quasicommutative semigroup. Then S is a commutative semigroup.*

Proof. Suppose that $S \times S$ is a quasicommutative semigroup. Evidently, S is quasicommutative. By way of contradiction, assume that S is not commutative. Then there exist elements a, b of S such that $ab \neq ba$. Theorem 1 implies that S is periodic Thus we have $a^k = e = e^2$ for a positive integer k It follows from (4) that $(a, a)(b, e) = (b, e)(a, e)^n$ for a positive integer n. Using Lemma we get $(a, a) . . (b, e) = (b, e)(a, a)$ and so $ab = ba$, which is a contradiction. Hence S is commutative.

Corollary 1. *Let S be a semigroup such that $S \times S \times S \times S$ is a quas hamiltonian semigroup. Then S is a commutative semigroup.*

Corollary 2. *The variety of all commutative semigroups is the largest variety of quasi hamiltonian semigroups.*

Corollary 3. *Let m, n be positive integers. Then every semigroup satisfying the identity $xy = y^n x^m$ is commutative.*

References

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