Josef Král
Note on generalized multiple Perron integral

Časopis pro pěstování matematiky, Vol. 110 (1985), No. 4, 371--374

Persistent URL: http://dml.cz/dmlcz/118252

Terms of use:

© Institute of Mathematics AS CR, 1985

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
NOTE ON GENERALIZED MULTIPLE PERRON INTEGRAL

JOSEF KRÁL, Praha
(Received March 31, 1984)

We show that every real-valued function which is GP-integrable in the sense of [2] must be Lebesgue measurable. Using this result we obtain a dominated convergence theorem for the GP-integral which answers the question posed in [2], Remark 3 (cf. also Remark 11 in [1]).

By an interval (in \( \mathbb{R}^m \)) we mean a Cartesian product of \( m \) closed one-dimensional intervals of positive length. Given such an interval \( I \) we choose a cube \( K \supseteq I \) of minimal volume and put

\[
\nu(I) = \frac{\text{vol} I}{\text{vol} K} ;
\]

if a point \( x \in I \) has been specified in \( I \), then the interval is termed a pointed interval and will be denoted by \((x, I)\). By a \( P \)-partition of an interval \( J \) we mean any finite system

\[
(x^1, I^1), \ldots, (x^p, I^p)
\]

of mutually non-overlapping pointed intervals whose union equals \( J \). If \( x = [x_1, \ldots, x_m] \in \mathbb{R}^m \) and \( \varrho > 0 \), then we adopt the notation

\[
B[x, \varrho] = \bigcap_{j=1}^m \langle x_j - \varrho, x_j + \varrho \rangle
\]

for the cube of side-length \( 2\varrho \) centered at \( x \). A positive function on \( J \) is called a gauge. If \( \delta \) is a gauge on \( J \), then the \( P \)-partition (1) is termed \( \delta \)-fine provided

\[
I^j \supseteq B[x^j, \delta(x^j)], \quad j = 1, \ldots, p .
\]

Let now \( J \) be a fixed interval and consider a real-valued function

\[
F : I \mapsto F(I)
\]

of an interval \( I \subseteq J \). Given \( x \in J, \alpha \in (0, 1) \) and \( \varrho > 0 \), we put

\[
s_F^\alpha(x) = \inf_{I \subseteq J} \frac{F(I)}{\text{vol} I} ,
\]

where \( I \subseteq J \) runs over all intervals satisfying

\[
x \in I \subseteq B[x, \varrho], \quad r(I) \geq \alpha ;
\]
further we define
\[ *F(x) = \sup_{\varepsilon > 0} *F_{\varepsilon}^b(x), \quad *F(x) = \inf_{0 < \varepsilon \leq 1} *F_{\varepsilon}(x), \]
\[ *F(x) = -(-F)(x). \]
If \(*F(x) = *F(x) \in \mathbb{R}, then \( F \) is said to be derivable at \( x \) and the common value of \(*F(x)\) and \(*F(x)\) is denoted by \( F'(x) \) and termed the derivative of \( F \) at \( x \).

Let us recall, for the case of real-valued functions, the definition of the GP-integral from [2].

**Definition.** We say that a real-valued function \( f \) on \( J \) is **GP-integrable over \( J \)** if there exists a real number \( k \) satisfying the following condition:

For any \( \varepsilon > 0 \) and \( \alpha \in (0, 1) \) there exists a gauge \( \delta \) on \( J \) such that
\[ |k - \sum_{j=1}^{p} f(x^j) \text{vol} I^j| < \varepsilon \]
holds for each \( \delta \)-fine \( P \)-partition (1) of \( J \) fulfilling
\[ r(I^j) \geq \alpha, \quad j = 1, \ldots, p. \]

The corresponding \( k \) is called the **GP-integral of \( f \) over \( J \)** and denoted by
\[ \text{GP} \int_J f. \]

**Remark 1.** Let us recall some basic facts established in [2].

The existence of the integral (4) guarantees that \( \text{GP} \int_J f \) exists for each interval \( I \subset J \) and
\[ I \mapsto \text{GP} \int_I f \]
is an additive function of an interval \( I \subset J \).

If \( f \) is a function on \( J \) with a convergent Lebesgue integral
\[ L \int_J f, \]
then the integral (4) exists as well and coincides with (6).

For later use let us rephrase Proposition 9 from [2] in the following form.

**Saks-Henstock lemma.** Let \( f \) be a real-valued function which is GP-integrable over \( J \) and suppose that \( \varepsilon > 0, \alpha \in (0, 1) \). If \( \delta \) is a gauge on \( J \) corresponding to \( \varepsilon \) and \( \alpha \) as in the above definition, then
\[ \left| \sum_{j=1}^{p} \left[ \text{GP} \int_{I^j} f - f(x^j) \text{vol} I^j \right] \right| < \varepsilon \]
holds for each finite system of mutually non-overlapping pointed intervals (1) in J fulfilling the conditions (2), (3).

Proof. If (1) is any system of non-overlapping pointed intervals in J satisfying (2), (3), then we can complete $I^1, \ldots, I^p$ by adding some intervals $I^{p+1}, \ldots, I^{p+q}$ so as to get a partition $I^1, \ldots, I^{p+q}$ of J formed by mutually non-overlapping intervals. If $r(I^n) < \alpha$ for some n, then $I^n$ can be further subdivided into non-overlapping intervals $I^n_i$ with $r(I^n_i) \geq \alpha$.

Finally, each interval I of the new partition which did not occur in the original system $\{I^1, \ldots, I^p\}$ can be replaced by its $\delta$-fine $P$-partition which is formed by intervals $\bar{I}$ similar to I, so that $r(\bar{I}) = r(I) \geq \alpha$. In such a way we arrive at a $\delta$-fine $P$-partition of J including the given system (1) and to this $P$-partition Proposition 9 from [2] applies.

**Theorem 1.** Let $f$ be GP-integrable over J. Then (5) is a function of an interval $I \subset J$ which is derivable at almost every $x \in J$ and its derivative coincides with $f$ a.e. in J; in particular, $f$ is Lebesgue measurable. If, moreover, the Lebesgue integral of $f$ over J exists, then it necessarily converges and (6) coincides with (4).

**Corollary.** For any non-negative real-valued function $f$ on J, the existence of the GP-integral (4) implies the convergence of the Lebesgue integral (6) and the equality of both.

Proof. Let us denote by $F$ the function of an interval $I \subset J$ defined by (5). Fix an arbitrary $\alpha \in (0, 1)$ and $\varrho > 0$ and consider the set

$$M_{\varrho} = \{x \in J; \quad *F_\alpha(x) \leq f(x) - 2\varrho\}.$$  

Admitting that the outer Lebesgue measure of $M_{\varrho}$ equals $2\sigma > 0$ we choose $\varepsilon > 0$ small enough to guarantee that

$$\varepsilon < \varrho \sigma.$$  

Now let $\delta$ be a gauge on J corresponding to $\varepsilon$ and $\alpha$ as in the above definition. Associating with each $x \in M_{\varrho}$ the system of all intervals $I \subset J$ satisfying the conditions

$$x \in I \subset B[x, \delta(x)], \quad r(I) \geq \alpha, \quad F(I) / \text{vol} I \leq f(x) - \varrho,$$

we obtain, as $x$ runs over $M_{\varrho}$, a system of intervals which covers $M_{\varrho}$ in the sense of Vitali. By Vitali’s covering theorem, there exists a finite disjoint subsystem of pointed intervals (1) satisfying (2), (3) such that

$$\sum_{j=1}^{p} \text{vol} I^j \geq \sigma.$$  

Employing the Saks-Henstock lemma we arrive at

$$\varepsilon > \sum_{j=1}^{p} [f(x^j) \text{vol} I^j - F(I^j)] \geq \varrho \sum_{j=1}^{p} \text{vol} I^j \geq \varrho \sigma$$  

373
which contradicts (7). Thus each $M_\varphi$ has vanishing Lebesgue measure and, in particular, the same is true for

$$M_\infty = \{ x \in J; \ast F_x = -\infty \} .$$

By Ward's theorem (cf. [3], p. 139), $F$ is derivable at almost all points in $J \setminus M_\infty$, i.e. almost everywhere in $J$. We have seen that the derivative satisfies the inequality

$$F' \geq f \text{ a.e. in } J .$$

Since $f$ may be replaced by $-f$, we have $F' = f$ a.e. in $J$ which means that $f$ is Lebesgue measurable (cf. Theorem (4.2) in [3], p. 112).

If $f \geq 0$, then the corresponding $F$ is a non-negative additive function of an interval whose derivative $F' (= f \text{ a.e.})$ is known to be Lebesgue summable (cf. Theorem (7.4) in [3], p. 119); consequently, (6) is convergent and coincides with (4).

If $f$ is of variable sign and its Lebesgue integral exists, then at least one of the functions $f^+ = \max (f, 0), f^- = \max (-f, 0)$ must have a convergent Lebesgue integral; let it be $f^+$. The equality $f^- = f^+ - f$ implies that $f^-$ is GP-integrable and, being non-negative, must also have a convergent Lebesgue integral.

As a consequence of Theorem 1 we get the following dominated convergence theorem for the GP-integral.

**Theorem 2.** Let $\{ f_n \}$ be a pointwise convergent sequence of GP-integrable functions over $J$. If there exist GP-integrable functions $g, h$ such that

$$g \leq f_n \leq h$$

on $J$ for all $n$, then $f = \lim f_n$ is also GP-integrable over $J$ and

$$\text{GP} \int_J f = \lim_{n} \text{GP} \int_J f_n .$$

**Proof.** We know that all the functions $f_n - g \geq 0$ are Lebesgue summable and are dominated by $h - g$ which is Lebesgue summable as well. As $n \to \infty, f_n - g \to f - g$ pointwise on $J$, whence it follows by the Lebesgue dominated convergence theorem that

$$\text{GP} \int_J f_n - \text{GP} \int_J g = L \int_J (f_n - g) \to L \int_J (f - g) = \text{GP} \int_J f - \text{GP} \int_J g .$$

**References**


*Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).*