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SOME REMARKS ON THE STRONG LIMIT-POINT CONDITION OF SECOND-ORDER LINEAR DIFFERENTIAL EXPRESSIONS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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1. INTRODUCTION

Let $\mathbb{R}$ denote the real field and let $[a, b)$ be a closed-open interval of $\mathbb{R}$ with $-\infty < a < b \leqq \infty$. Let C denote the complex field; if $\lambda \in C$ we write $\lambda = \mu + i\nu$. The function spaces of complex-valued locally integrable and locally absolutely continuous functions on $[a, b)$ are denoted respectively by $L_{\text{loc}}[a, b)$ and $AC_{\text{loc}}[a, b)$.

Let p, q and w be given coefficients satisfying the following basic conditions

(1.1) (i) $p, q, w\colon [a, b) \to \mathbb{R}$ and are Lebesgue measurable
(ii) $p(x) > 0$ (almost all $x \in [a, b)$) and $p^{-1}(\equiv 1/p) \in L_{\text{loc}}[a, b)$
(iii) $q \in L_{\text{loc}}[ab)$
(iv) $w(x) > 0$ (almost all $x \in [a, b)$) and $w \in L_{\text{loc}}[a, b)$.

In this paper we are concerned with properties of the symmetric linear quasi-differential equation (the so-called generalised Sturm-Liouville equation)

$$-(py')' + qy = \lambda wy \quad \text{on} \quad [a, b) \tag{1.2}$$

or, equivalently, the symmetric linear quasi-differential expression

$$w^{-1}((-pf')' + qf) \quad \text{on} \quad [a, b)\,. \tag{1.3}$$

Here, and to follow, a prime $'$ denotes classical differentiation.

In (1.2) a solution $y\colon [a, b) \to C$ and both y and $py' \in AC_{\text{loc}}[a, b)$; similarly for f in (1.3).

Let $L^2_w[a, b)$ denote the Lebesgue function space of complex-valued measurable functions f satisfying

$$\int_a^b w(x)\,|f(x)|^2\,\mathrm{d}x \equiv \int_b^a w|f|^2 < \infty\,.$$

The original classification of (1.2), equivalently (1.3), as in the limit-point or limit-circle condition at b in $L^2_w[a, b)$ is due to Weyl [13]; see also Titchmarsh [12].

The strong limit-point and Dirichlet conditions for (1.2) and (1.3) were named about the time of the paper by Everitt, Giertz and Weidmann [7], although earlier results are also significant. See also the results in Everitt, Giertz and McLeod [6]; Kalf [11]; Everitt [3]; Everitt and Wray [9]. In particular the work of Kalf [11] is important for the results of this present paper.

As far as notations and definitions are concerned the most suitable reference is Everitt [4, section 3.1]. The Green's formula for both (1.2) and (1.3) may be written as, for all $[\alpha, \beta] \subset [a, b)$

$$(1.4) \qquad \int_\alpha^\beta \{\bar{g}\, M[f] - \overline{M}[g] f\} = [fg](\beta) - [fg](\alpha)$$

valid for all $f, g, pf', pg' \in AC_{\text{loc}}[a, b)$ where

$$(1.5) \qquad M[f] := -(pf')' + qf \quad \text{on} \quad [a, b)$$

$$(1.6) \qquad [fg](x) := (p\bar{g}' . f - \bar{g} . pf')(x) \quad (x \in [a, b))\,.$$

Similarly the Dirichlet formula takes the form

$$(1.7) \qquad \int_\alpha^\beta \{p\bar{g}' . f' + q\bar{g}f\} = p\bar{g}' . f\Big|_\alpha^\beta + \int_\alpha^\beta \overline{M}[g] f\,.$$

Let $\Delta \subset L^2_w[a, b)$ denote the linear manifold defined by

$$(1.8) \qquad \Delta := \{f\colon [a, b) \to C \mid \text{(i) } f \text{ and } pf' \in AC_{\text{loc}}[a, b) \quad \text{(ii) } f \text{ and } w^{-1} M[f] \in L^2_w[a, b)\}\,.$$

The following definitions are then made; see [4, section 3.1]:

(i) the differential expression M is limit-point (LP) at the end-point b in $L^2_w[a, b)$ if

$$(1.9) \qquad \lim_{\beta \to b-} [fg](\beta) = 0 \quad (\text{all } f, g \in \Delta)$$

(ii) M is strong limit-point (SLP) at b in $L^2_w[a, b)$ if

$$(1.10) \qquad \lim_{\beta \to b-} (p\bar{g}' . f)(\beta) = 0 \quad (\text{all } f, g \in \Delta)$$

(iii) M is Dirichlet (D) at b in $L^2_w[a, b)$ if

$$(1.11) \qquad p^{1/2} f' \quad \text{and} \quad |q|^{1/2} f \in L^2[a, b) \quad (\text{all } f \in \Delta)\,.$$

The LP condition (1.9) is motivated by consideration of the Green's formula (1.4). Similarly the SLP and D conditions (1.10 and 11) are connected with the Dirichlet formula (1.7).

The LP classification of the differential expression M at b in $L^2_w[a, b)$, and so of the differential equation (1.2), is dependent only on the coefficients p, q and w; see [4, section 3.1]. The papers [11], [3] and [9] are particularly concerned with finding conditions on these coefficients to place M in both the SLP and D conditions at b in $L^2_w[a, b)$; this classification of M is particularly important in applications.

The first theorem presented below overlaps in part with the main theorem of Kalf [4]. However a separate proof is given here since the details given are required in the proof of subsequent results, which are not included in the Kalf proof.

We start a proof of a lemma due to Kalf, see [11, page 204]; the proof given here is more appropriate for the methods of this paper.

Lemma. *Let the coefficients p, q and w satisfy basic conditions* (i) *to* (iv) *of* (1.1). *Suppose that M is Dirichlet at the end-point b in $L^2_w[a, b)$, and that at least one of p^{-1}, q, w is not in $L[a, b)$; then M is strong limit-point at b in $L^2_w[a, b)$.*

Proof. See section 3 below.

The first theorem depends essentially on the conditions which prevail when the weight coefficient w, which belongs to $L_{loc}[a, b)$, is not in $L[a, b)$.

Theorem 1. *Let the coefficients p, q and w satisfy the basic conditions* (i) *to* (iv) *of* (1.1). *Suppose that*

(1) *there exists a non-negative number A such that*

$$q(x) + A\,w(x) \geqq 0 \quad (\textit{almost all } x \in [a, b)) \tag{2.1}$$

(2) $w \notin L[a, b)$.

Then M is Dirichlet and strong limit-point at b in $L^2_w[a, b)$.

Proof. See section 4 below.

The second theorem gives a result whether w belongs to or does not belong to $L[a, b)$; however, in view of theorem 1, the result is more significant when $w \in L[a, b)$.

Theorem 2. *Let the coefficients p, q and w satisfy the basic conditions* (i) *to* (iv) *of* (1.1). *Let $q_\pm [a, b) \to \mathbb{R}_+$ be defined by*

$$q_\pm(x) = \tfrac{1}{2}\{|q'(x)| \pm q(x)\} \quad (\textit{all } x \in [a, b))\,.$$

Suppose that

(1) *there exist a positive number k and a non-negative number A such that*

$$p(x)\,q_+(x) \geqq k^2 > 0 \quad \textit{and} \quad q_-(x) \leqq A\,w(x) \tag{2.2}$$

both hold for almost all $x \in [a, b)$, and

(2) $$\int_a^b w(x) \exp\left[2k \int_a^x p^{-1}\right] dx = \infty .$$

Then M is Dirichlet and strong limit-point at b in $L^2_w[a, b)$.

Proof. See section 5 below .

Notes to theorem 2. (i) The condition (2.2) on q_- can be relaxed, see Kalf [11, page 199], but the requirement given here is appropriate for this paper.

(ii) Note that (2) implies at least one of w and p^{-1} is not in $L[a, b)$.

(iii) We shall show in an example that $k > 0$ cannot be replaced by $k \geqq 0$; if $k = 0$ then we may have the limit-circle case at b.

Corollary (to theorem 2). *Let $b = \infty$ and $p(x) = 1$ for all $x \in [a, \infty)$; let q and w satisfy conditions* (iii) *and* (iv) *of* (1.1). *Suppose that*

(1) $$q(x) \geqq k^2 > 0 \ (\text{almost all } x \in [a, \infty))$$

(2) $$\int_a^\infty e^{2kx} w(x) \, dx = \infty .$$

Then the differential expression $-y'' + qy$ on $[a, \infty)$ is Dirichlet and strong limit-point at ∞ in $L^2_w[a, \infty)$.

Proof. This follows at once from theorem 2.

This corollary is useful in examples.

3. PROOF OF THE LEMMA

Since the coefficients p, q and w are real-valued on $[a, b)$ it is sufficient to prove that (1.10) holds, i.e. $\lim_b p\bar{g}' . f = 0$, for all real-valued f, g in Δ.

Given that M is D at b it follows from the Dirichlet formula (1.7) that $\lim_b p\bar{g}' . f$ exists and is finite. Suppose then M is not SLP at b; then there exist real-valued f and g in Δ with

(3.1) $$\lim_{x \to b} (pg'f)(x) = \mu \neq 0 .$$

Without loss of generality we can suppose that $\mu > 0$, and then, for some $\alpha \in [a, b)$, $f(x) > 0$ for all $x \in [\alpha, b)$. This implies $pg' > \frac{1}{2}\mu/f$ in some $[\alpha, b)$, and so $|pf'g'| > > \frac{1}{2}\mu|f'|/f$ on $[\alpha, b)$. Integrating over $[\alpha, b]$

(3.2) $$\int_\alpha^\beta |pf'g'| \geqq \tfrac{1}{2}\mu \int_\alpha^\beta |f'|/f \geqq \tfrac{1}{2}\mu\left| \int_\alpha^\beta f'/f \right| = \tfrac{1}{2}\mu|\ln(f(\beta)) - \ln(f(\alpha))| .$$

Since M is D at b it now follows, for d, D with $0 < d < D < \infty$, that

$$0 < d \leqq f(x) \leqq D < \infty \quad (x \in [\alpha, b)). \tag{3.3}$$

Suppose now either q or $w \in L[a, b)$; then since both $|q|\,|f|^2$ and $w|f|^2 \in L[a, b)$ we have a sequence $\{\beta_n: n = 1, 2, \ldots\}$ with $\{\beta_n\} \to b$ and $\{f(\beta_n)\} \to 0$ as $n \to \infty$. However this contradicts (3.3) and hence (3.1); thus M must be SLP at b.

Suppose now both q and $w \in L[a, b)$ then from the conditions of the lemma $p^{-1} \notin L[a, b)$. We have, since M is D at b,

$$\int_a^b p^{-1}|pg'|^2 = \int_a^b p|g'|^2 < \infty$$

and so for a sequence $\{\beta_n\}$, as above, we have $\{(pg')(\beta_n)\} \to 0$ as $n \to \infty$. But from (3.3) this implies $\{(pg'f)(\beta_n)\} \to 0$ as $b \to \infty$ and so, from (3.1), it follows that $\mu = 0$. This is a contradiction and again M must be SLP at b.

4. PROOF OF THEOREM 1

Again it is sufficient to prove that M is D at b by proving that (1.11) holds for all real-valued $f \in \Delta$.

From the Dirichlet formula (1.7) and (1) of (2.1)

$$\int_\alpha^\beta \{pf'^2 + (q + Aw)f^2\} = pf'f\Big|_\alpha^\beta + \int_\alpha^\beta w \,.\, w^{-1} M[f]f + A\int_\alpha^\beta wf^2 \tag{4.1}$$

where the integrand on the left is non-negative on $[a, b)$. Suppose the integral on the left tends to ∞ as $\beta \to b-$; then, since $f \in \Delta$, both integrals on the right remain finite and so $\lim_{b-} pf'f = \infty$. Hence, for some $\alpha \in [a, b)$, $pf'f > 0$ on $[\alpha, b]$; from (ii) of (1.1) it then follows that $f'f > 0$ almost everywhere on $[\alpha, b)$, i.e. f^2 is monotonic increasing on $[\alpha, b)$. However $f \in L^2_w[a, b)$ and, from (2) of (2.1), $w \notin L[a, b)$ and so there is a sequence $\{\beta_n\} \to b$ with $\{f(\beta_n)\} \to 0$. This gives a contradiction and the integral on the left of (4.1) must remain finite as $\beta \to b-$ and this implies that M is D at b in $L^2_w[a, b)$.

Finally M is SLP at b in $L^2_w[a, b)$ from the lemma of section 2.

5. PROOF OF THEOREM 2

We begin by noting that from the second part of (2.2)

$$-q_- \geqq -Aw\,, \quad \text{i.e.} \quad q = q_+ - q_- \geqq -Aw\,, \quad \text{i.e.} \quad q + Aw \geqq 0$$

on $[a, b)$; hence condition (1) of theorem 1 is satisfied. If now $w \notin L[a, b)$, and certainly (2) of theorem 2 then holds, all the conditions of theorem 1 are satisfied and M is D at b. Thus without loss of generality we can suppose that

$$w \in L[a, b) \quad \text{and} \quad p^{-1} \notin L[a, b), \tag{5.1}$$

the latter to hold in order to ensure that (2) is satisfied.

As before it is necessary to prove only that (1.11) is satisfied for all real-valued f in Δ. From Dirichlet's formula (1.7) we obtain, using also $q = q_+ - q_-$,

$$\int_a^x \{pf'^2 + q_+ f^2\} = pf'f\Big|_a^x + \int_a^x w \,.\, w^{-1} M[f] f + \int_a^x q_- f^2 \tag{5.2}$$

valid for all $x \in (a, b)$. Using the second part of (2.2) and with $f \in \Delta$, it follows that both integrals on the right of (5.2) are bounded as $x \to b$. If the integral on the left, which has a non-negative integrand, is unbounded as $x \to b$ then $(pf'f)(x) \to \infty$ as $x \to b$. Thus for some $\mu > 0$ and for $y \in (a, b)$ we have $pf'f > \mu > 0$ on (y, b); without loss of generality we may assume $f(x) > 0$ for all $x \in (y, b)$. Hence $f'f \geqq \mu p^{-1}$ for almost all $x \in (y, b)$ and so f^2, and also f, is monotonic increasing on (y, b).

Integrating over $[y, x]$ gives

$$f(x)^2 \geqq 2\mu \int_y^x p^{-1} + f(y)^2 \geqq 2\mu \int_y^x p^{-1} \quad (x \in [y, b)). \tag{5.3}$$

Thus from (5.1) it follows that $\lim_b f = \infty$. Now choose $\alpha \in (y, b)$ so that $f(\alpha) > 1$; then from (5.3) we obtain

$$f(x)^{-2} \leqq \frac{1}{2\mu} \left\{ \int_y^x p^{-1} \right\}^{-1} \quad (x \in [\alpha, b)).$$

Squaring this result and integrating over $[\alpha, \beta]$ gives

$$\begin{aligned} 4\mu^2 \int_\alpha^\beta (pf^4)^{-1} &\leqq \int_\alpha^\beta \frac{1}{p(x)} \left\{ \int_y^x p^{-1} \right\}^{-2} \mathrm{d}x = \\ &= \left[- \left\{ \int_y^x p^{-1} \right\}^{-1} \right]_\alpha^\beta < \left\{ \int_y^\alpha p^{-1} \right\}^{-1} < \infty \end{aligned} \tag{5.4}$$

valid for all $\beta \in [\alpha, b)$.

Returning to (5.2) we obtain, on using $2ab \leqq a^2 + b^2$ for $a, b \geqq 0$ and the first of (2.2)

$$2k \int_a^x f'f \leqq 2 \int_a^x (pq_+)^{1/2} f'f \leqq \int_a^x \{pf'^2 + q_+ f^2\} \leqq (pf'f)(x) + O(1)$$

valid for all $x \in (a, b)$. Here $0(\cdot)$ is the standard notation. Integrating the first term gives

$$k f(x)^2 \leqq (pf'f)(x) + O(1) \quad (x \in (a, b)). \tag{5.5}$$

Now let $x \in (\alpha, b)$, divide (5.5) by $(pf^4)(x)$ and integrate over $[\alpha, \beta]$ to give

$$(5.6) \qquad k\int_\alpha^\beta (pf^2)^{-1} \leqq \int_\alpha^\beta f'f^{-3} + O\left(\int_\alpha^\beta (pf^4)^{-1}\right) \leqq f(\alpha)^{-2} + O(1)$$

for all $\beta \in [\alpha, b]$, on using (5.4).

Now divide (5.5) by pf^2 and integrate over $[\alpha, \beta]$ to give for all $x > \alpha$ (recall $f(x) > f(\alpha) > 1$)

$$k\int_\alpha^x p^{-1} \leqq \ln(f(x)) - \ln(f(\alpha)) + O\left(\int_\alpha^x (pf^2)^{-1}\right) \leqq \ln(f(x)) + L$$

where, from (5.6), L is a positive constant. Taking exponentials gives, for a positive K,

$$\exp\left[k\int_\alpha^x p^{-1}\right] \leqq K f(x) \quad (x \in [\alpha, b))$$

and squaring and integrating over $[\alpha, \beta]$

$$\int_\alpha^\beta w(x) \exp\left[2k\int_\alpha^x p^{-1}\right] \mathrm{d}x \leqq K^2 \int_\alpha^\beta w(x) f(x)^2 \, \mathrm{d}x$$

valid for all $\beta \in [\alpha, b)$. Since $f \in L^2_w[a, b)$ this last result is a contradiction on condition (2) of theorem 2.

Thus both $p^{1/2}f'$ and $q_+^{1/2}f \in L^2[a, b)$. From the second part of (2.2) it follows that $q_-^{1/2}f \in L^2[a, b)$ for all $f \in \Delta$. Hence, when (5.1) is the case, we also have M is D in $L^2_w[a, b)$.

Finally M is SLP at b in $L^2_w[a, b)$ from the lemma of section 2.

6. SOME EXAMPLES

We discuss here only two examples. Reference to other examples should be made to Kalf [11], and Everitt and Wray [9, sections 3 and 5].

The first example is not covered by the results in [9] or [11] but does come under the corollary to theorem 2. This example also illustrates, in one sense, the best possible nature of the result in theorem 2 in that the lower bound k, for the product pq_+, cannot be improved. Let $a = 0$, $b = \infty$ $p(x) = 1$ $q(x) = \nu^2$ $w(x) = e^{-2x}$ $(x \in [0, \infty))$ where the number $\nu \geqq 0$. We see that all the conditions of the corollary are satisfied if we take $k = \nu$ and $\nu \geqq 1$. The resulting differential equation (1.2) in this case is D and SLP at ∞ in $L^2_w[0, \infty)$. Explicitly we have

$$-y''(x) + \nu^2 y(x) = \lambda e^{-2x} y(x) \quad (x \in [0, \infty))$$

which has solutions $Z_\nu(e^{-x}\nu\lambda)$ where Z_ν is any Bessel function of order ν. When ν is not a positive integer we have solutions

$$J_\nu(e^{-x}\sqrt{\lambda}) \sim K(\nu, \lambda)\, e^{-\nu x}, \quad J_{-\nu}(e^{-x}\sqrt{\lambda}) \sim L(\nu, \lambda)\, e^{\nu x}$$

as $x \to \infty$; when $\nu = 1$ we have

$$J_1(e^{-x}\sqrt{\lambda}) \sim K(\lambda)\, e^{-x}, \quad Y_1(e^{-x}\sqrt{\lambda}) \sim L(\lambda)\, e^{x}$$

again as $x \to \infty$. These results show that this example is LP at ∞ in $L^2_w[0, \infty)$ when $\nu \geqq 1$, and LC at ∞ in $L^2_w[0, \infty)$ when $0 \leqq \nu < 1$. The requirement $\nu \geqq 1$ for the corollary to hold is seen to match the actual classification of the equation.

The second example is

$$a = 0, \quad b = \infty \quad p(x) = 1 \quad q(x) = k^2 \geqq 0 \quad w(x) = x^{-4} \exp[-2x^{-1}].$$

This example is regular at 0 but singular at ∞. The conditions of the corollary are satisfied as long as $k > 0$, but not when $k = 0$. This example also comes under the theorem in [11]; again the conditions required in [11] cannot be satisfied when $k = 0$. It is not known if solutions of the resulting differential equation can be obtained explicitly in terms of known transcendental functions when $k > 0$; this would seem unlikely. However when $k = 0$ Halvorsen [10] has shown that solutions may be obtained in terms of Bessel functions of order zero; in fact independent solutions are

$$x\, J_0(\exp[-x^{-1}]\sqrt{\lambda}), \quad x\, Y_0(\exp[-x^{-1}]\sqrt{\lambda}).$$

An analysis then shows that the equation is LC at ∞ in $L^2_w[0, \infty)$, i.e. when $k = 0$. When $k > 0$ both [11] and this paper show that the equation is D and SLP at ∞ in $L^2_w[0, \infty)$. See also Everitt and Halvorsen [8].

7. THE TITCHMARSH-WEYL *m*-COEFFICIENT

The results in this paper have applications to the theory of the *m*-coefficient for the differential equation (1.2). See Bennewitz and Everitt [2, section 8] and the forthcoming paper [1] which is a revision of the work in [2].

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