Časopis pro pěstování matematiky, roč. 111 (1986), Praha

EQUATIONS OF MAGNETOHYDRODYNAMICS: PERIODIC SOLUTIONS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

(Received May 15, 1985)

1. INTRODUCTION

The aim of this paper is to prove the existence of time-periodic solutions to a system of equations appearing in magnetohydrodynamics. The treated system of equations is a bit non-standard in the respect that the usually adopted form of the Ohm law is substituted here by one in which the terms describing Hall's effect and ion slip are allowed for [6, p. 30]. In this respect the present paper generalizes paper [7], where the Ohm law was applied in its simplest form.

Various existence results have been proved in numerous papers out of which we quite here only [2]–[4], [9]–[11], [7], [8] since these papers have some common points with the present paper. A more detailed description can be found in [7].

In this paper, equally as in [10], [7] and [8], the displacement current is not neglected. As said above the Ohm law includes terms reflecting Hall's effect and ion slip.

We investigate the system of equations governing the velocity v of viscous electrically conducting incompressible fluid, the magnetic field B and the electric field E in the fluid. The fluid is supposed to occupy a bounded region $\Omega \subset \mathbb{R}^3$ homeomorphic to a ball. The boundary $\partial\Omega$ of Ω is supposed to be sufficiently smooth and perfectly conducting.

The system of equations and boundary conditions is the following:

$$\varrho(v_t + (v, \nabla)\, v - \nu\, \Delta v) = -\nabla p + \varrho\tilde{f} + qE + j \times B\,, \tag{1.1}$$

$$\operatorname{div} v = 0\,, \tag{1.2}$$

$$v(\cdot, x) = 0 \quad \text{for} \quad x \in \partial\Omega\,, \tag{1.3}$$

$$B_t + \operatorname{rot} E = 0\,, \tag{1.4}$$

$$\operatorname{div} B = 0\,, \tag{1.5}$$

$$B_n(\cdot, x) = 0 \quad \text{for} \quad x \in \partial\Omega\,, \tag{1.6}$$

$$\varepsilon E_t + j - \operatorname{rot} B/\mu = \tilde{j}\,, \tag{1.7}$$

$$\varepsilon \operatorname{div} E = q\,, \tag{1.8}$$

$$E_\tau(\cdot, x) = 0 \quad \text{for} \quad x \in \partial\Omega\,, \tag{1.9}$$

$$j = \sigma(E + v \times B) + \alpha_1(x)\, j \times B + \alpha_2(x)\,(j \times B) \times B\,. \tag{1.10}$$

Supposing $\tilde{f}$ and $\tilde{j}$ are 2π-periodic in t and small we shall prove that there exist 2π-periodic functions (v, p, B, E) satisfying (1.1)–(1.10) (Theorem 3.1). This is a perturbation result obtained by applying the contraction mapping principle when the existence results for linearized equations have been given.

Now we recall the original meaning of the variables involved:

v velocity of fluid,
$\tilde{f}$ given external mass force,
p pressure,
B and E magnetic and electric field vectors,
j current density vector,
$\tilde{j}$ given external electric current density vector,
q net charge density.

The quantities ϱ(density), ν(viscosity), ε(permitivity), μ(permeability) and σ(electrical conductivity) are supposed to be positive constants. The functions α_1 and α_2 are supposed to be smooth on $\bar{\Omega}$ and for technical reasons we have to suppose

$$\alpha_1(x) = 0 \quad \text{for} \quad x \in \partial\Omega\,. \tag{1.11}$$

The subscripts n and τ denote, respectively, the normal and tangential components of a vector, i.e. if $n(x)$ denotes the unit outward normal to $\partial\Omega$ at a point x and "$\cdot$" the scalar product in $\mathbb{R}^3$, then

$$B_n(t, x) = (B(t, x) \,.\, n(x))\, n(x)\,, \quad E_\tau(t, x) = E(t, x) - E_n(t, x)\,.$$

In the next section, Section 2, the system in question will be reduced to a more suitable one. In Section 3, the spaces will be defined and the result presented. In Section 4 we shall state the results for linearized equations. In the last section, Section 5, the main result will be proved.

2. AN EQUIVALENT SYSTEM

To transform (1.1)–(1.10) to a more suitable set of equations we now define two operators V and W.

Firstly, given a function $a \in H^k(\Omega)$, the space of scalar functions on Ω with square integrable generalized derivatives up to order k, we set $Va = \operatorname{grad} \varphi$, where φ satisfies $\Delta\varphi = a$ in Ω and $\varphi = 0$ and $\partial\Omega$. Hence, for every positive k, V is a linear bounded operator from $H^{k-1}(\Omega)$ into $(H^k(\Omega))^3$ satisfying $(Va)_\tau = 0$ on $\partial\Omega$.

Secondly, to define the operator W we begin by introducing two spaces, $k \geqq 1$ integer,

$$\mathring{J}_\tau^k(\Omega) = \{u \in (H^k(\Omega))^3,\ \operatorname{div} u = 0,\ u_\tau = 0 \text{ on } \partial\Omega\},$$

$$\mathring{J}_n^k(\Omega) = \{u \in (H^k(\Omega))^3;\ \operatorname{div} u = 0,\ u_n = 0 \text{ on } \partial\Omega\},$$

By [4], the operator rot is a homeomorphism of $\mathring{J}_\tau^{k+1}(\Omega)$ onto $\mathring{J}_n^k(\Omega)$. The inverse operator is denoted by W, i.e., given $B \in \mathring{J}_n^k(\Omega)$, we denote by WB the function $w \in \in (H^{k+1}(\Omega))^3$ satisfying

$$\operatorname{rot} w = B,\quad \operatorname{div} w = 0 \text{ in } \Omega,$$

$$w_\tau = 0 \quad \text{on} \quad \partial\Omega.$$

In what follows we denote

$$a(t) = \operatorname{div} E(t).$$

Then, suppressing for simplicity the dependence on t of the functions involved, we have in virtue of (1.4) and (1.9), for every t fixed,

$$\begin{aligned} \operatorname{rot}(E - Va) &= -B_t && \text{in } \Omega, \\ \operatorname{div}(E - Va) &= 0 && \text{in } \Omega, \\ (E - Va)_\tau &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Since by (1.5) and (1.6), $\operatorname{div} B_t(t, \cdot) = 0$ in Ω and $B_n(t, \cdot) = 0$ on $\partial\Omega$, we can write

$$E(t) = Va(t, \cdot) - WB_t(t, \cdot).$$

Setting

$$J = j - \sigma(E + v \times B),$$

we get from (1.10)

$$\begin{aligned} (2.1) \qquad & J - \alpha_1(x)\, J \times B - \alpha_2(x)\,(J \times B) \times B = \\ & = \sigma\, \alpha_1(x)\, D \times B + \sigma\, \alpha_2(x)\,(D \times B) \times B, \end{aligned}$$

where

$$D = E + v \times B.$$

Viewing (for a while) D and B as elements of $\mathbb{R}^3$ we find by applying the Implicit Function Theorem a smooth function ι mapping $\mathcal{O}_6 \times \bar{\Omega}$, $\mathcal{O}_6$ a neighbourhood of 0 in $\mathbb{R}^6$, into $\mathbb{R}^3$, a neighbourhood $\mathcal{O}_3$ of 0 in $\mathbb{R}^3$, and positive $\varkappa_1, \varkappa_2$ such that $J = = \iota(B, D, x)$ is the only $J \in \mathcal{O}_3$ satisfying (2.1) provided $|B| \leqq \varkappa_1$, $|D| \leqq \varkappa_2$ and $x \in \bar{\Omega}$. Here $|B|$ means $(\sum_{j=1}^{3} B_j^2)^{1/2}$. Assuming $D_\tau(x) = 0$ for $x \in \partial\Omega$, we easily see, as a consequence of (1.11), that $J_\tau = 0$. Evidently, $\iota(0, D, x) = 0$.

Using the operators defined above we can reduce the system (1.1)–(1.10) to an equivalent one which has turned out to be more suitable for the intended investigation. The system we shall now be interested in is the following:

$$\varrho(v_t - \nu\,\Delta v) = \varrho(v, \nabla)\,v - \nabla p + \varepsilon a(Va - WB_t) + \tag{2.2}$$

$$+ \left[J(v, B, a) + \sigma(Va - WB_t + v \times B)\right] \times B + \varrho\tilde{f},$$

$$\operatorname{div} v = 0, \tag{2.3}$$

$$v(t, \cdot) = 0 \quad \text{on} \quad \partial\Omega, \tag{2.4}$$

$$\varepsilon\mu B_{tt} + \sigma\mu B_t + \operatorname{rot}\operatorname{rot} B = \mu \operatorname{rot}\left(J(v, B, a) + \sigma(v \times B) - \tilde{j}\right), \tag{2.5}$$

$$\operatorname{div} B = 0, \tag{2.6}$$

$$B_n(t, \cdot) = 0, \quad \operatorname{rot}_\tau B(t, \cdot) = 0 \quad \text{on} \quad \partial\Omega, \tag{2.7}$$

$$\varepsilon a_t + \sigma a = -\operatorname{div}\left(J(v, B, a) + \sigma(v \times B) - \tilde{j}\right), \tag{2.8}$$

where

$$J(v, B, a) = \iota(B, Va - WB_t + v \times B, x). \tag{2.9}$$

and $\operatorname{rot}_\tau B$ stands for $(\operatorname{rot} B)_\tau$.

In this system the functions $\tilde{f}$ and $\tilde{j}$ are supposed to be periodic functions of the variable t. For simplicity we shall suppose all functions in question to be 2π-periodic in t.

Moreover, we shall assume $\tilde{j}_\tau(t, \cdot) = 0$ on $\partial\Omega$ for all t.

3. SPACES AND RESULT

Periodic functions — as a consequence of their 2π-periodicity in the time variable — will be considered as given on $S^1 \times \Omega$, S^1 the unit circle. Since most functions will be given on $S^1 \times \Omega$, this symbol will be omitted in the notation of spaces. Only in case we want to emphasize that the function involved is considered as depending on the spatial variables only we append Ω explicitly.

As in Section 1 we use the subscripts n, τ or 0 to denote the vector functions having zero normal component, zero tangential component or just zero on the boundary $\partial\Omega$, respectively.

The space of functions on $S^1 \times \Omega$ which have square integrable generalized derivatives up to order k, $k \in N$, will be denoted by H^k. This space is equipped with the usual norm,

$$\|u\|_k = \Big(\sum_{|\alpha| \leq k} \|D^\alpha_{t,x} u\|_0^2\Big)^{1/2},$$

where

$$\|u\|_0 = \left(\int_0^{2\pi} \int_\Omega |u(t, x)|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2},$$

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \quad \text{and} \quad D^\alpha_{t,x} = D^{\alpha_0}_t D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} D^{\alpha_3}_{x_3}.$$

In particular, we have $H^0 = L^2(S^1 \times \Omega)$. Besides H^k, we shall use the following spaces of functions defined on $S^1 \times \Omega$, $k \in N$:

$$X^k = \{u;\ D_t^\alpha D_{t,x}^\beta D_{t,x}^\gamma u \in H^0 \text{ for } \alpha \leqq 1,\ 2\beta_0 + |\beta_1| \leqq 2,\ |\gamma| \leqq k\}\,,$$
$$Y^k = \{u;\ D_t^\alpha D_{t,x}^\beta u \in H^0 \quad \text{for} \quad \alpha \leqq 1,\ |\beta| \leqq k\}\,,$$
$$Z^k = \{u;\ D_x^\alpha D_{t,x}^\beta u \in H^0 \text{ for } |\alpha| \leqq 1,\ |\beta| \leqq k\}\,.$$

The norms in these spaces are defined in an obvious manner to guarantee the completeness of all these spaces.

The same notations will be used to denote the spaces of 3-dimensional vector functions on $S^1 \times \Omega$. The subscript s will stand for solenoidal functions, i.e. functions with divergence equal to zero.

Thus, for example, $Y_{s,\tau}^k$, $k \geqq 1$, is the space consisting of all real 3-dimensional vector functions $v \in Y^k$ with $\operatorname{div} v(t, \cdot) = 0$ in Ω and $v_\tau(t, \cdot) = 0$ on $\partial\Omega$ for all t.

As usual, by P we denote the orthogonal projection of $(L^2(\Omega))^3$ onto $\mathring{J}_n^0$, the completion in $(L^2(\Omega))^3$ of all solenoidal functions from $(C_0^\infty(\Omega))^3$. The projection in $L^2(S^1 \times \Omega)$ defined by $v(t, \cdot) \to Pv(t, \cdot)$ will be denoted by P as well.

Now we are in a position to give the result of the paper.

Theorem 3.1. *Let $k \in \mathbb{N}$, $k \geqq 3$, $\tilde{f} \in Y^k$ and $\tilde{j} \in Z_\tau^{k+1}$. Then, under the assumption that $\|\tilde{f}\|_{Y^k}$ and $\|\tilde{j}\|_{Z^{k+1}}$ are both sufficiently small, there exist functions $v \in X^k$, $B \in H^{k+2}$, $a \in H^{k+1}$ and p with $\nabla p \in Y^k$ such that (2.2)–(2.9) are satisfied.*

Remark 3.1. Since the systems (1.1)–(1.10) and (2.2)–(2.9) are equivalent we also get a solution to the original problem.

4. LINEARIZED EQUATIONS

We introduce three operators representing the linear parts of the equations (2.2), (2.5) and (2.8). Firstly, we set

$$L_1 v = v_t - \nu P\,\Delta v\,.$$

Lemma 4.1. *Let $f \in Y^k$. Then there exists a unique $v \in X_{s,0}^k$ satisfying $L_1 v = f$. Moreover, denoting v by $\Lambda_1 f$ we have $\|\Lambda_1 f\|_{X^k} \leqq c\|f\|_{Y^k}$.*

Remark 4.1. The function v from this lemma is a solution to the problem

$$v_t - \nu P\,\Delta v = Pf \quad \text{in} \quad S^1 \times \Omega\,,$$
$$v(t, \cdot) = 0 \quad \text{on} \quad S^1 \times \partial\Omega\,.$$

This means that there is a function p such that

$$\nabla p \in Y^k \quad \text{and} \quad v_t - \nu\,\Delta v = f - \nabla p\,.$$

Secondly, for $\varepsilon > 0$ we set

$$L_2^\varepsilon B = \varepsilon\mu B_{tt} + \sigma\mu B_t + \operatorname{rot}\operatorname{rot} B\,.$$

Lemma 4.2. *Let $g \in Y^k_{s,n}$. Then there is a unique $B \in H^{k+2}_{s,n}$ with $(\operatorname{rot} B)_\tau = 0$ on $S^1 \times \partial\Omega$ such that $L_2^\varepsilon B = g$. Moreover, denoting B by $\Lambda_2^\varepsilon g$, we have $\|\Lambda_2^\varepsilon g\|_{k+2} \leqq \leqq c\|g\|_{Y^k}$ with a constant c independent of ε.*

Remark 4.2. The assumptions on g could be slightly weakened. If we suppose $g \in H^k_{s,n}$ and $D_t^{k+1} g \in H^0$ Lemma 4.2 also holds.

Thirdly, for $\varepsilon > 0$ we set

$$L_3^\varepsilon a = \varepsilon a_t + \sigma a\,.$$

Lemma 4.3. *Let $h \in H^k$. Then there is a unique $a \in Y^k$ such that $L_3^\varepsilon a = h$. Moreover, denoting a by $\Lambda_3^\varepsilon h$, we have*

$$\varepsilon\|(\Lambda_3^\varepsilon h)_t\|_k + \|\Lambda_3^\varepsilon h\|_k \leqq c\|h\|_k$$

with a constant independent of ε.

Proofs of all these three lemmas go along standard lines [cf. 7]. The corresponding results for stationary problems are applied in the proof. In the case of Lemma 4.1 this is the problem

$$-\Delta v = -\nabla p + f\,, \quad \operatorname{div} v = 0 \text{ in } \Omega\,, \quad v = 0 \text{ on } \partial\Omega\,,$$

which is investigated in [2] and [12]. In the case of Lemma 4.2 we rely on the following result from [4], cf. also [1]. For $h \in \mathring{J}^k_n(\Omega)$, $k \geqq 0$, there exists a unique $w \in H^{k+2}(\Omega)$ satisfying

$$\operatorname{rot}\operatorname{rot} w = h\,, \quad \operatorname{div} w = 0 \text{ in } \Omega\,,$$

$$w_n = 0\,, \quad \operatorname{rot}_\tau w = 0 \text{ on } \partial\Omega\,.$$

5. PROOF OF THEOREM 3.1

Applying the operator P to (2.2) and using Λ_1, Λ_2^ε and Λ_3^ε from Lemmas 4.1–4.3 in (2.2), (2.5) and (2.8) we get the following system of equations for v, B and a:

(5.1) $$v = \varrho^{-1}\Lambda_1 Pf(v, B, a)\,,$$

(5.2) $$B = \mu\Lambda_2^\varepsilon g(v, B, a)\,,$$

(5.3) $$a = -\Lambda_3^\varepsilon h(v, B, a)\,,$$

where

$$f(v, B, a) = \varrho(v, \nabla) v + \varepsilon a(Va - WB_t) +$$

$$+ [J(v, B, a) + \sigma(Va - WB_t + v \times B)] \times B + \varrho\tilde{f},$$

$$g(v, B, a) = \operatorname{rot}(J(v, B, a) + \sigma(v \times B) - \tilde{j}),$$

$$h(v, B, a) = \operatorname{div}(J(v, B, a) + \sigma(v \times B) - \tilde{j})$$

and $J(v, B, a)$ is given by (2.9).

Let us set $\tilde{H}^k_{s,n} = \{B \in H^k_{s,n};\ \operatorname{rot}_\tau B = 0 \text{ on } S^1 \times \partial\Omega\}$ and $M^k = X^k_{s,0} \times \tilde{H}^{k+2}_{s,n} \times H^{k+1}$. The space M^k equipped with an obvious norm $|||\cdot|||$ is a Banach space. For any integer k, $k \geqq 2$, we can find $R > 0$ such that the ball $M^k_R = \{u \in M^k;\ |||u||| < R\}$ is mapped by J into Z^{k+1}. This follows from the Moser lemma on a mapping given by a composition operator [5]. By definition of Z^{k+1}, to prove $J(v, B, a) \in Z^{k+1}$ means to show that $D^\alpha_x J(v, B, a) \in H^{k+1}$ for any α, $|\alpha| \leqq 1$. Applying the chain rule we find that $D^\alpha_x J(v, B, a)$ has the form $\phi(x, w_1, \ldots, w_p)$, where ϕ is a smooth function of all arguments and $w_j = w_j(t, x)$ are elements of H^{k+1} with $\|w_j\|_{k+1} \leqq C_R$, a constant depending on R. By Sobolev's lemma we have $\|w_j\|_C \leqq c_S c_R$ and thus by Moser's lemma $\phi(\cdot, w_1, \ldots, w_p) \in H^{k+1}$, which means that $J(v, B, a) \in Z^{k+1}$ in our case. By a similar argument we show that J is a Lipschitzian map of M^k_R into Z^{k+1}.

Repeating the argument for $k \geqq 3$, we can prove that (f, g, h) is a Lipschitzian mapping of M^k_R into $Y^k \times Y^k \times H^{k+1}$. Since $(J(v, B, a) + \sigma(v \times B) - \tilde{j})_\tau = 0$ on $S^1 \times \partial\Omega$, we have by [1, Note 2, p. 51] $g(v, B, a) \in H^{k+1}_{s,n}$. Applying the contraction mapping principle we obtain the desired result.

Remark 5.1. The fact that in proving the above result we have used only estimates which do not depend on ε makes it possible to let ε tend to 0 and get the solution of the system of equations obtained formally when $\varepsilon = 0$, i.e. when the displacement current is cancelled and the net charge is set equal to zero.

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