José Luis Palacios

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Časopis pro pěstování matematiky, Vol. 111 (1986), No. 4, 337--339

Persistent URL: http://dml.cz/dmlcz/118278

ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY

Vydává Matematický ústav ČSAV, Praha

SVAZEK 111 * PRAHA 28. 11. 1986 * ČÍSLO 4

A CONJECTURE ON A CLASS OF MATRICES

José Luis Palacios, Caracas

(Received June 27, 1983)

Let $k(A)$ be the number of positive elements in a non-negative square matrix A. It is conjectured that for the class of connected matrices, $k(A^n)$ is a non-decreasing function of n. If this result is true, it generalizes a result of Šidák's in [3].

1.

Let A be an irreducible and aperiodic (= primitive) $M \times M$ matrix having at most one zero on its main diagonal. We shall call such A, a Šidák matrix. In [3], Z. Šidák proved that $k(A^n)$ is a non-decreasing function of n whenever A is a Šidák matrix. We conjecture that the same is true for the connected matrices defined below.

Denote $S = \{1, \ldots, M\}$. The members of S are called states. Let $I \subset S$ and let $F(I)$ be the set of such indices $j \in S$ that $a_{ij} > 0$ for some $i \in I$ (a_{ij} is the (i, j)-entry of A).

Definition. An $M \times M$ matrix is *connected* if

(i) it is irreducible,

(ii) for any proper nonempty subset I of S, $F(I) \cap F(I^c) \neq \emptyset$.

Connected matrices arise in at least two setups: they are used by Seneta in the estimation of non-negative matrices from marginal totals (see [2]; Seneta's definition differs slightly from ours, he substitutes (i) by the condition that A have no zero row or column). Also, in [1] it is proved, among other things, that if A is the transition matrix of a homogeneous Markov chain, and $\mathscr{E}$ is the exchangeable sigma-field generated by the chain, $\mathscr{E}$ is trivial iff A is connected ($\mathscr{E}$ is the sigma-field of events which are invariant under finite permutations of their coordinates).

2.

The definition in 1 can be restated in terms of the following equivalence relation:

Definition. We say that two states i and j are *neighbors* if there exists a state k such that $a_{ki}a_{kj} > 0$. We say that i and j are *equivalent* (notation $i \sim j$) if there is a sequence of states $i = i_1, i_2, \ldots, i_n = j$ such that i_k and i_{k+1} are neighbors for $k = 1, \ldots, n-1$.

With this definition we have:

Proposition. ([2], 2.2) *An* $M \times M$ *matrix is connected iff*

(i) *it is irreducible,*

(ii) *all states* $1, \ldots, M$ *are equivalent.*

Another way to picture a connected matrix A is to think of the entries of A as the squares of a chessboard. Then A is connected iff it is irreducible and a rook can move on the board to every row and column visiting only those entries which are positive.

We prove now:

Proposition. Any Šidák matrix is a connected matrix.

Proof. Assume that M is the only state such that $a_{MM} = 0$. We will prove that $i \sim M$ for any state i. Since A is irreducible, there are states $i = i_0, i_1, \ldots, i_n = M$ such that $a_{i_{k-1}, i_k} > 0$ for $1 \leqq k \leqq n$. From this, and from the fact that $a_{i_k, i_k} > 0$ for $0 \leqq k \leqq n$ we conclude that i_{k-1} and i_k are neighbors for $1 \leqq k \leqq n$, and $i \sim M$.

In case that $a_{kk} > 0$ for all $1 \leqq k \leqq M$, we single out any state to play the role of the state M in the argument above.

3. FINAL COMMENTS

Of (i) and (ii) in the definition of a connected matrix A, only (ii) seems to be essential for $k(A^n)$ to be non-decreasing in all the examples that we have analyzed. Look at these examples with $M = 4$:

$$A = \begin{bmatrix} + & + & + & + \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} + & + & + & + \\ 0 & 0 & 0 & + \\ 0 & 0 & 0 & + \\ 0 & 0 & 0 & + \end{bmatrix} \quad C = \begin{bmatrix} 0 & + & + & 0 \\ 0 & 0 & + & + \\ + & 0 & 0 & + \\ + & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}$$

It is easily verified that

$$\begin{aligned} k(A^n) &= 16 \quad \text{for} \quad n \geqq 2, & k(B^n) &= 7 \quad \text{for} \quad n \geqq 2, \\ k(C^2) &= 10, & k(D^2) &= 9, \\ k(C^3) &= 14, & k(D^n) &= 10 \quad \text{for} \quad n \geqq 3, \\ k(C^n) &= 16 \quad \text{for} \quad n \geqq 4. \end{aligned}$$

Notice that all A, B, C and D have the minimum number of positive entries for a matrix to have the rook property (ii), namely $2M - 1$, but, while A and C are irreducible and hence connected, B and D are not. Irreducibility is the ingredient that turns a matrix with the rook property into a primitive matrix (see [1], p. 26 for a proof that connectivity implies primitivity), so that there is an integer n such that $k(A^n) = M^2$.

For an arbitrary primitive $M \times M$ matrix, let γ be the minimum integer such that $k(A^\gamma) = M^2$. It is known ([2], p. 58) that $\gamma \leqq M^2 - 2M + 2$ and the bound is sharp. For connected matrices, however, γ seems to be much smaller than this bound, and that is another problem to be studied.

Note added in proof. The following example due to Chris Parrish (U. Simón Bolívar) shows that the conjecture fails to be true in case condition (i) of irreducibility is dropped.

If $A = \begin{bmatrix} + & + & + & 0 \\ 0 & 0 & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & 0 \end{bmatrix}$ then $k(A^2) = 9$ and $k(A^3) = 8$.

References

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Author's address: Universidad Simón Bolívar, Departamento de Matemáticas, Apartado 80659, Caracas, Venezuela.