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Betweenness spaces and tree algebras

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By a \textit{betweenness space} we mean a pair \((X, \beta)\), where \(X\) is a nonvoid set, and \(\beta \subseteq X^3\) is a ternary relation on \(X\) subject to the following axioms:  \(^1\)

\begin{align*}
\beta_1 & : \beta_{abb}, \\
\beta_2 & : \beta_{aba} \Rightarrow a = b, \\
\beta_3 & : \beta_{abc} \Rightarrow \beta_{cba}, \\
\beta_4 & : \beta_{abc} \land \beta_{acd} \Rightarrow \beta_{bcd}, \\
\beta_5 & : \beta_{abc} \land \beta_{bcd} \land b \neq c \Rightarrow \beta_{abd}.
\end{align*}

Here, \(\beta_{xyz}\) means that \(y\) lies between \(x\) and \(z\). If, for every \(a, b\), there is only a finite number of elements between \(a\) and \(b\), we call the space \((X, \beta)\) discrete.

The betweenness relation \(\beta\) may be called connected, or linear, whenever the additional condition

\(L: \beta_{abc} \lor \beta_{bca} \lor \beta_{eab}\)

is also fulfilled. The axiom system \(\beta_2 - \beta_5, L\) for linear betweenness has appeared already in \([1], [4]\), where it is proved that each of these axioms is independent of the others and that \(\beta_1\) follows from \(L\) and \(\beta_2\). Clearly, \(\beta_1\) cannot be derived from the conditions \(\beta_2 - \beta_5\) alone, hence, our axiom system is also independent. In \([2]\) we considered betweenness spaces in which \(\beta\) fulfils, instead of \(L\), two weaker conditions of smoothness

\(IS: \beta_{acb} \land \beta_{adb} \Rightarrow \beta_{acd} \lor \beta_{adbc},\)
\(OS: \beta_{abc} \land \beta_{abd} \Rightarrow \beta_{acd} \lor \beta_{adbc},\)

both being consequences of \(\beta_2 - \beta_5, L\) (cf. \([4]\)). Here we shall deal with spaces in which, in addition to \(\beta_1 - \beta_5\), the following axiom for \(\beta\) is valid:

\(M: \exists x(\beta_{axb} \land \beta_{bxo} \land \beta_{cxa}).\)

We call these spaces \(M\)-spaces. Obviously, \(\beta_1\) is a consequence of \(M\) and \(\beta_2\). It will be shown that the notion of an \(M\)-space is equivalent to that of a tree algebra \([3], [5]\), and, using a result from \([5]\), a one-to-one correspondence between discrete \(M\)-spaces and trees will be established.

\(^1\) Here, as well as throughout the whole paper, we omit the universal quantifiers which might be placed in front of a formula to bound the free variables occurring in it.
In what follows, let \((X, \beta)\) be a fixed betweenness space, if not otherwise stated. Elements of \(X\) will be referred to as points. For brevity, we write \(xyz\) for \(\beta xyz\). Those properties of \(\beta\) that will be needed below are summarized in the following lemma, where \(\mu \subset X^4\) is a relation on \(X\) defined by

\[
\mu_{abep} \iff apb \land bpe \land cpa.
\]

**Lemma.** For arbitrary \(a, b, c, p, q \in X\) we have:

- \(\mu_{abep} \iff \mu_{baep} \iff \mu_{acbp},\)
- \(\mu_{abcb} \iff abc,\)
- \(\mu_{abep} \land cdp \Rightarrow \mu_{abdp},\)
- \(\mu_{abep} \land bcd \land c \neq p \Rightarrow \mu_{abdp},\)
- \(\mu_{abep} \land eqp \land bq \land p \neq q \Rightarrow \mu_{abdp},\)
- \(\mu_{abep} \land abd \land acd \Rightarrow p = b \lor p = c.\)

If \(\beta\) fulfils the condition \(M\), then \(IS\) holds and, moreover, the following implications are valid:

- \(\mu_{7}: \mu_{abep} \land bde \Rightarrow apd,\)
- \(\mu_{8}: \mu_{abep} \land \mu_{abeq} \Rightarrow p = q,\)
- \(\mu_{9}: \mu_{abep} \land \mu_{abdp} \land apd \Rightarrow \mu_{acqp},\)
- \(\mu_{10}: \mu_{abep} \land \mu_{abdp} \land p \neq q \Rightarrow \mu_{bedp} \lor \mu_{bcdp}.\)

**Proof.** \(\mu_1\) and \(\mu_2\) are obvious.

- \(\mu_3: \text{cpa} \land cdp \Rightarrow dpa\)
  \(\land \text{cpb} \land cdp \Rightarrow dpb\)
  \(\land \text{apb} \land dpa \land dbp \Rightarrow \mu_{abdp}.\)
- \(\mu_4: \text{bpc} \land bcd \Rightarrow pcd\)
  \(\land \text{apc} \land pcd \land c \neq p \Rightarrow apd\)
  \(\land \text{bpc} \land pcd \land c \neq p \Rightarrow bp\)
  \(\land \text{apb} \land bpd \land apd \Rightarrow \mu_{abdp}.\)
- \(\mu_5: \mu_{abep} \land eqp \Rightarrow \mu_{abep}\)
  \(\mu_{abep} \land bq \land p \neq q \land \mu_{abdp}.\)
- \(\mu_6: \text{abd} \land apb \Rightarrow pb\)
  \(\land \text{cpb} \land pb \Rightarrow p = b \lor cp\)
  \(\land \text{apc} \land aed \Rightarrow p\)
  \(\land \text{cpd} \land pcd \Rightarrow p = c \lor dp\)
  \(\land \text{dpd} \land pcd \Rightarrow p = c.\)
- **IS:** \(\mu_{acd\ell} \chi_0\)
  \(\mu_{acd\ell} \land acb \land adb \Rightarrow x_0 = c \lor x_0 = d\)
  \(\mu_{acd\ell} \land (x_0 = c \lor x_0 = d) \Rightarrow aed \lor aed.\)
- \(\mu_7: \text{bpc} \land bdc \Rightarrow bpd \lor bdp\)
  \(\text{bdp} \land bpa \Rightarrow apd\)
  \(\text{bdp} \land bde \Rightarrow pdc\)
  \(\text{pdc} \land apc \Rightarrow apd.\)
\[ \mu 8: \mu abc p \land b q c \Rightarrow a p q \quad [\mu 7] \\
\mu abc q \land b p c \Rightarrow a q p \quad [\mu 7] \\
apq \land a q p \Rightarrow p = q \lor a p a \quad [\beta 3, \beta 5] \\
apa \land a q p \Rightarrow p = q. \quad [\beta 2, \beta 2] \\
\mu 9: \mu bad q \land a p d \Rightarrow b q p \\
\mu a e b p \land b q p \Rightarrow \mu a c q p. \quad [\mu 7] \\
\mu 10: a p b \land a q b \Rightarrow a p q \lor a q p \\
\mu d a b q \land a p q \land b p c \land p \neq q \Rightarrow \mu b c d q \\
\mu c b a p \land a q p \land b q p \land p \neq q \Rightarrow \mu b c d p. \quad [\mu 5, \mu 1] \\
\]

We say that \( p \) is the median of the points \( a, b, c \), if \( p \) is the unique point that satisfies the condition \( \mu abc p \). From \( \mu 8 \) we get

**Corollary.** \( (X, \beta) \) is an M-space if and only if every three points of \( X \) have the median.

Following [3], we call a pair \((X, m)\) a tree algebra, if \( m : X \rightarrow X \) is a ternary operation on \( X \) which satisfies the following axioms (we write \((xyz)\) for \( m(xyz)\)):

\[ m 1: (a a b) = a, \]
\[ m 2: (a b c) = (b c a) = (a c b), \]
\[ m 3: ((a b c) b d) = (a b c b d), \]
\[ m 4: (a b d) \neq (b c d) \neq (a c d) \Rightarrow (a b d) = (a c d). \]

Then the operation \( m \) is said to be a median operation. As in [5], we omit the condition (explicit in [3]) that \( X \) must be finite. Note that \( m 4 \) may be rewritten in the form

\[ m 4': (a b d) = (b c d) \lor (b c d) = (a c d) \lor (a b d) = (a c d). \]

Any median operation \( m \) has the following properties:

\[ m 5: ((a b c) b c) = (a b c), \quad [m 3, m 2, m 1] \]
\[ m 6: (a c d) = (b c d) \Rightarrow (a b c) = (a b d). \]

For \( m 6 \) see [3], Theorem 1.3. Now we shall prove the main

**Theorem.** Let \( m \) be a ternary operation, and let \( \beta \) be a ternary relation on \( X \). Then

a) if \((X, m)\) is a tree algebra, and if \( \beta \) is defined by

\[ (\ast) \quad \beta a b c \iff m(abc) = b, \]

then \((X, \beta)\) is an M-space, and the condition

\[ (\ast \ast) \quad m(abc) = p \iff \beta a p b \land \beta b p c \land \beta c p a \]

holds;

b) if \((X, \beta)\) is an M-space, and if \( m \) is defined by \((\ast \ast)\), then \((X, m)\) is a tree algebra, and \( \beta \) fulfills \((\ast)\).

**Proof.** (a) Assume \( m \) is a median operation and \( \beta \) fulfills \((\ast)\). Then \( \beta 1 - \beta 3 \) easily follow from \( m 1 \) and \( m 2 \). Furthermore, if \( (abc) = b \) and \( (acd) = c \), then
$$\text{(bed) = ((abc) cd) = (bc(aed)) = (bce) = c;}$$

hence, $\beta 4$ is valid. To prove $\beta 5$, assume that $(abc) = b$, $(bed) = c$, $b \neq c$. Then $(abd) \neq c$, for otherwise, owing to $m5$, we should have

$$b = (abc) = (ab(abd)) = (abd) = c.$$ 

Hence, $(ace) \neq (edc) \neq (add)$, and, in virtue of $m4$, $(abd) = b$. To prove $M$, let $x_0 = (abc)$. Then by $m5$, $(ax_0 b) = x_0$, $(bx_0 c) = x_0$, $(cx_0 a) = x_0$. Finally, $(**)$ now means that

$$(abc) = p \iff (apb) = (bpc) = (cpa) = p.$$ 

By $m5$ the left hand equality implies the right hand ones. The converse equality follows from $m6$: if $(apb) = (bpc)$; then $(abc) = (cpa) = p$.

(b) Assume $\beta$ is a betweenness and $m$ fulfils $(**$). Let us check that $m1$--$m4$ and $(*)$ are valid. By $\mu 1$, $\mu 2$ we have $\mu abbb$, hence, by $\mu 8$, $\mu abbp$ implies $p = b$, and $m1$ follows. $m2$ means that

$$\mu abcp \wedge \mu bcaq \wedge \mu acbr \Rightarrow p = q = r,$$

and this is true in virtue of $\mu 1$ and $\mu 8$. To prove $m3$, we need to show that

$$\mu abcp \wedge \mu pbdq \wedge \mu cbdr \wedge \mu abrs \Rightarrow q = s.$$ 

If $p = q$, then

$$\mu abcp \Rightarrow \mu bcaq \quad [\mu 1]$$

$$\mu bcaq \wedge \mu cbdr \wedge bqp \Rightarrow \mu barq \quad [\mu 9]$$

$$\mu barq \wedge \mu abrs \Rightarrow q = s. \quad [\mu 8]$$

If $r = s$, then

$$\mu cbdr \Rightarrow \mu cbds \quad [\mu 1]$$

$$\mu cbds \wedge \mu bcap \wedge bsa \Rightarrow \mu bdps \quad [\mu 9]$$

$$\mu bdps \wedge \mu pbdq \Rightarrow q = s. \quad [\mu 8]$$

If $p \neq q$ and $r \neq s$, then

$$\mu dbpq \wedge bpa \Rightarrow \mu dbaq \quad [\mu 4]$$

$$\mu abrs \wedge brd \Rightarrow \mu abds \quad [\mu 4]$$

$$\mu dbaq \wedge \mu abds \Rightarrow q = s. \quad [\mu 1, \mu 8]$$

Furthermore, $m4$ means that

$$\mu abdp \wedge \mu bcdq \wedge \mu acdr \wedge p \neq q \wedge q \neq r \Rightarrow p = r.$$ 

But we have

$$\mu abdp \wedge \mu acdr \wedge p \neq r \Rightarrow \mu bcdp \vee \mu bcr \quad [\mu 10]$$

$$\mu bcdp \wedge \mu bcdq \Rightarrow p = q \quad [\mu 1, \mu 8]$$

$$\mu bcr \wedge \mu bcdq \Rightarrow r = q. \quad [\mu 1, \mu 8]$$

Finally, $(*)$ coincides with $\mu 2$.

Therefore, there is a one-to-one correspondence between $M$-spaces and tree algebras. In [5], such a correspondence is established between the so called discrete tree algebras and trees. This result includes the finite as well as infinite case,
and is a generalization of a result in [3] for finite trees. The resulting correspondence between discrete M-spaces and trees may be explicitly described as follows. Let \((X, E)\) be a tree, where \(X\) is the set of its vertices and \(E\) is the set of edges. Let \(\beta abc\) mean that there is a path in the tree from \(a\) to \(c\) passing through \(b\). Then \((X, \beta)\) is a discrete M-space. Vice versa, if \((X, \beta)\) is such a space and 

\[ E = \{(a, b) \in X^2 : a \neq b \land \forall x(\beta axb \Rightarrow a = x \lor x = b)\} \]

then \((X, E)\) is a tree.

Added November 5, 1984. In the meantime, several papers, in which ternary spaces and/or ternary algebras are discussed, have appeared. We comment here three of them being more or less closely connected with our main subject. The class of ternary spaces considered in [6] includes our betweenness spaces and, hence, M-spaces as well. Furthermore, every tree algebra is a medium in the sense of [6]. Theorem 2.1 [6] asserts that any medium is a ternary space, and Proposition 3.5 shows when a discrete ternary space is the ternary space of a medium. Some results on tree algebras are contained in Sect. 6 of [7]; this paper has also a valuable bibliography. In [8], a theorem from [5] is disproved concerning independence of a certain system of conditions on segments in tree algebras.

The author is indebted to the referee for indicating two inaccuracies in the proof of Lemma.

References
