Časopis pro pěstování matematiky

Vladimír Palko

On the Lebesgue decomposition of Gleason measures

Časopis pro pěstování matematiky, Vol. 112 (1987), No. 1, 1--5

Persistent URL: `http://dml.cz/dmlcz/118291`

ON THE LEBESGUE DECOMPOSITION OF GLEASON MEASURES

VLADIMÍR PALKO, Bratislava
(Received August 1, 1983)

Summary. In the article the notions of dominancy and singularity of signed states on a logic are defined. A theorem is proved, which asserts that if ω and m are finite signed states on the logic of all closed subspaces of a separable Hilbert space, ω is nonnegative, then m can be written as a sum of two signed states m_1, m_2 such that m_1 is dominated by ω and m_2 is singular to ω. An example is given which shows that in distinction to the case of measures on a σ-algebra in this case the nonnegativity of m does not guarantee the nonnegativity of m_1 and m_2.

Keywords: Signed states on a logic, dominancy and singularity of signed states on a logic, decomposition of signed states on a logic.
AMS classification: 81 B

The notions of dominancy and singularity of signed measures are defined for example in [5]. The known theorem on the Lebesgue decomposition asserts that if μ is a σ-finite measure and ν a σ-finite signed measure on a σ-algebra $\mathscr{S}$, then there exist two signed measures ν_1, ν_2 such that $\nu = \nu_1 + \nu_2$, ν_1 is dominated by μ and ν_2 is singular to μ. If ν is a measure, then ν_1, ν_2 are also measures. In the present paper we introduce the notions of dominancy and singularity for signed states on a logic. The structure of a general logic does not enable us to formulate for signed states any theorem similar to that on the Lebesgue decomposition. Therefore we deal with a special logic $\mathscr{L}(H)$, which consists of closed subspaces of a complex or real separable Hilbert space H, $\dim H \geqq 3$. Finite signed states on $\mathscr{L}(H)$ are also called Gleason measures. We prove a theorem which asserts that if ω is a nonnegative finite signed state and m a finite signed state on $\mathscr{L}(H)$, then m can be written as a sum of two signed states m_1, m_2, where m_1 is dominated by ω and m_2 is singular to ω. On the other hand, we show that there is a difference between the classical result for measures on a σ-algebra and that for signed states on $\mathscr{L}(H)$, because there are nonnegative finite signed states on $\mathscr{L}(H)$ whose dominated and singular parts are not nonnegative.

Let $\mathscr{L}$ be a σ-lattice with the first and last element 0 and 1, respectively, and an orthocomplementation $\perp: a \mapsto a^\perp$, $a, a^\perp \in \mathscr{L}$, which satisfies

(i) $(a^\perp)^\perp = a$ for all $a \in \mathscr{L}$,

(ii) if $a \leqq b$, then $b^\perp < a^\perp$,

(iii) $a \vee a^\perp = 1$ for all $a \in \mathscr{L}$,

(iv) $a \leqq b$ implies $b = a \vee (a^\perp \wedge b)$.

A σ-lattice $\mathscr{L}$ fulfilling all the above conditions is called a logic. Two elements a, b of $\mathscr{L}$ are called orthogonal (written $a \perp b$) if $a \leqq b^\perp$. A signed state on a logic $\mathscr{L}$ is a map m from $\mathscr{L}$ into $R \cup \{\infty\} \cup \{-\infty\}$ such that

$$\text{(i)} \quad m(0) = 0\,, \qquad \text{(ii)} \quad m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)\,,$$

where a_i, $i = 1, 2, \ldots$, is any sequence of pairwise orthogonal elements of $\mathscr{L}$.

Definition 1. Let m be *a signed state and* ω *a nonnegative signed state on* $\mathscr{L}$. Then m is said to be *dominated* by ω (written $m \ll \omega$), if $\omega(a) = 0$ implies $m(a) = 0$ for all $a \in \mathscr{L}$. m is said to be *singular* to ω (written $m \perp \omega$), if there exists an element $a_0 \in \mathscr{L}$ such that $a \leqq a_0$ implies $\omega(a) = 0$ and $b \leqq a_0^\perp$ implies $m(b) = 0$.

The logic $\mathscr{L}(H)$ is the most important case of logic. A deep theorem due to Gleason asserts that there is a one-to-one correspondence between nonnegative finite signed states on $\mathscr{L}(H)$ and nonnegative hermitean operators of the trace class on H. Gleason [4] proved that every finite nonegative signed state m is of the form $m(M) = \operatorname{tr} TP^M$, $M \in \mathscr{L}(H)$, where T is a nonnegative hermitean operator of the trace class and P^M is the projector corresponding to M. A hermitean operator T is of the trace class, if there exists an orthonormal basis $\{\varphi_i\}$ such that $\sum_i (|T|\varphi_i, \varphi_i) < \infty$. Then the sum $\operatorname{tr} T = \sum_i (T\varphi_i, \varphi_i)$ is called the trace of T and is independent of the basis used [6]. Dvurečenskij [2] generalized that theorem and proved that m is a finite signed state on $\mathscr{L}(H)$ if and only if m is of the form $m(M) = \operatorname{tr} TP^M$, where T is a hermitean operator of the trace class.

Theorem 1. *Let m be a finite signed state and ω a nonnegative finite signed state on $\mathscr{L}(H)$. Then there exist signed states m_1 and m_2 such that $m = m_1 + m_2$, $m_1 \ll \omega$, $m_2 \perp \omega$.*

Proof. We shall prove this theorem for the case when H is complex. The proof in the real case is a little simpler. Let S and T be the operators corresponding to ω and m, respectively. Let $\{\varphi_i\}$ be the orthonormal system of eigenvectors of S and $\{\lambda_i\}$ the sequence of the (positive) corresponding eigenvalues. Adding a suitable orthonormal system $\{\psi_j\}$ to the system $\{\varphi_i\}$ one obtains a complete orthonormal system in H. Since $\operatorname{tr} TP^M$ and $\operatorname{tr} SP^M$ are independent of the basis, we shall use the system $\{\varphi_i\} \cup \{\psi_j\}$. Hence

$$\omega(M) = \sum_i \lambda_i (P^M \varphi_i, \varphi_i)\,,$$
$$m(M) = \sum_i (TP^M \varphi_i, \varphi_i) + \sum_j (TP^M \psi_j, \psi_j)\,, \quad M \in \mathscr{L}(H)\,.$$

Denote $v_i(M) = (TP^M \varphi_i, \varphi_i)$. Let M_n, $n = 1, 2, \ldots$, be a sequence of pairwise orthogonal elements of $\mathscr{L}(H)$ and $M = \bigvee_{n=1}^{\infty} M_n$. A known proposition from the theory of Hilbert spaces asserts

$$P^M \varphi = \sum_{n=1}^{\infty} P^{M_n} \varphi\,, \quad \varphi \in H\,.$$

Using this fact and the continuity of T and the scalar product one obtains

$$v_i(M) = \sum_{n=1}^{\infty} v_i(M_n)\,.$$

Hence v_i is a σ-additive function, which is generally complex. Denote $\tilde{v}_i(M) = \operatorname{Re} v_i(M)$. Clearly, $\tilde{v}_i$ is also σ-additive. Thus it is a finite signed state. Denote $m_1(M) = \sum_i \tilde{v}_i(M)$, $M \in \mathscr{L}(H)$.

Let us assume that the number of φ_i is infinite,

$$\text{i.e.} \quad m_1 = \sum_{i=1}^{\infty} \tilde{v}_i\,.$$

(Of course, the other case is again simpler.) Then

$$m_1 = \lim_{n \to \infty} s_n\,, \quad \text{where} \quad s_n = \sum_{i=1}^{n} \tilde{v}_i\,.$$

A finite limit of a convergent sequence of finite signed states is also a signed state. Different proofs of this generalization of the known Nikodym's theorem were given in [1] and [3]. Hence m_1 is a signed state. Denote $m_2(M) = \sum_j \operatorname{Re}(TP^M\psi_j, \psi_j)$. It may be proved in the same way that m_2 is also a signed state. Obviously, $m_1(M) + m_2(M) = m(M)$, because $m(M)$ is real for all $M \in \mathscr{L}(H)$. Let us now prove

$$m_1 \ll \omega\,, \quad m_2 \perp \omega\,.\ \omega(M) = \sum_i \lambda_i \|P^M \varphi_i\|^2 = 0$$

implies $P^M\varphi_i = 0$ for all i. Thus $v_i(M) = 0$ for all i. Hence $m_1(M) = 0$. $m_1 \ll \omega$ is proved. Denote by N the closed subspace generated by the system $\{\varphi_i\}$. $M \subset N$ implies ψ_j is orthogonal to M for all j, hence $m_2(M) = 0$. $M \subset N^{\perp}$ implies φ_i is orthogonal to M for all i, hence $\omega(M) = 0$. Thus $m_2 \perp \omega$. Theorem is proved.

Now we give an example of nonnegative finite signed states ω and m on $\mathscr{L}(H)$ such that m is not decomposable into two nonnegative signed states m_1, m_2, $m_1 \ll \omega$, $m_2 \perp \omega$.

Example 1. Let $H = R^3$, $\varphi = (1, 0, 0)$, $\psi = (1, 1, 0)$, $\omega(M) = (P^M\varphi, \varphi)$, $m(M) = (P^M\psi, \psi)$, $M \in \mathscr{L}(H)$. Let us assume that $m = m_1 + m_2$, $m_1 \ll \omega$, $m_2 \perp \omega$, $m_1 \geqq 0$, $m_2 \geqq 0$. Then m_1 is necessarily of the form $m_1(M) = k\,\omega(M)$, where k is a positive real constant. Hence $m_2(M) = (P^M\psi, \psi) - k(P^M\varphi, \varphi)$. Let N be the onedimensional subspace generated by the vector $(1, -1, 0)$. Then $m_2(N) = -k(P^N\varphi, \varphi) < 0$. This is a contradiction with the assumption $m_2 \geqq 0$.

The following theorem characterizes the situation when the singular and the dominated part of a nonnegative Gleason measure are nonnegative.

Theorem 2. *Let ω, m be nonnegative finite signed states on $\mathscr{L}(H)$ and S, T the operators corresponding to ω, m, respectively. Then the following statements are equivalent:*

(i) *There exist nonnegative finite signed states m_1, m_2 such that $m = m_1 + m_2$, $m_1 \ll \omega$, $m_2 \perp \omega$.*

(ii) *T can be written as a sum of two nonnegative hermitean trace class operators T_1, T_2 such that $\mathscr{N}(T_2)^\perp \subset \mathscr{N}(S) \subset \mathscr{N}(T_1)$, where $\mathscr{N}(S)$, $\mathscr{N}(T_1)$, $\mathscr{N}(T_2)$ denote the null-spaces of S, T_1, T_2, respectively.*

Proof. (i) $\Rightarrow$ (ii) Let T_1, T_2 be nonnegative operators corresponding to m_1, m_2, respectively. The correspondence between the operators and the signed states is linear, hence $T = T_1 + T_2$. Let $\varphi \in \mathscr{N}(S)$. Then φ is orthogonal to all eigenvectors of S. Hence $\omega([\varphi]) = 0$, where $[\varphi]$ is the onedimensional subspace generated by φ. $m_1 \ll \omega$ implies $m_1([\varphi]) = 0$. Hence φ is orthogonal to all eigenvectors of T_1. An immediate consequence is $T_1\varphi = 0$. $\mathscr{N}(S) \subset \mathscr{N}(T_1)$ is proved. $\omega \perp m_2$ implies that there exists a subspace N such that $\omega(N) = 0$ and $m_2(N^\perp) = 0$. This yields $N \subset \mathscr{N}(S)$ and $N^\perp \subset \mathscr{N}(T_2)$. Hence $\mathscr{N}(T_2)^\perp \subset \mathscr{N}(S)$.

(ii) $\Rightarrow$ (i) Let m_1, m_2 be the signed states corresponding to T_1, T_2, respectively. Obviously, m_1, m_2 are nonnegative and $m = m_1 + m_2$. Using a similar consideration as in the proof of the converse proposition we obtain $m_1 \ll \omega$, $m_2 \perp \omega$.

References

[1] *T. A. Cook:* The Nikodym-Hahn-Vitale-Saks theorem for states on a quantum logic, Math. foundations of quantum theory (Proc. Conf., Loyola Univ., New Orleans, 1977) p. 275—286.

[2] *A. Dvurečenskij:* Signed states on a logic. Math. slovaca, *28* (1978), 33—40.

[3] *A. Dvurečenskij:* On convergences of signed states. Math. slovaca *28* (1978), 289—295.

[4] *A. M. Gleason:* Measures on the closed subspaces of a Hilbert space. J. Math. Mech. *6* (1957), 885—893.

[5] *P. R. Halmos:* Measure theory, New York 1950.

[6] *R. Schatten:* Norm ideals of completely continuous operators, Springer 1970.

Súhrn

ON THE LEBESGUE DECOMPOSITION OF GELASON MEASURES

VLADIMÍR PALKO

V článku sa definuje pojem dominovanosti a singulárnosti zovšeobecnených stavov na kvantovej logike. Dokazuje sa veta, ktorá hovorí, že ak ω a m sú konečné zovšeobecnené stavy na logike uzavretých podpriestorov separabilného Hilbertovho priestoru, pričom ω je nezáporný, potom m je súčtom dvoch zovšeobecnených stavov m_1 a m_2 takých, že m_1 je dominovaný ω a m_2 je singulárny na ω. Je daný príklad, že na rozdiel od klasického prípadu mier na σ-algebre nezápornosť m ešte nezaručuje nezápornosť m_1 a m_2.

Резюме

О РАЗЛОЖЕНИИ ЛЕБЕГА МЕР ГЛИСОНА

VLADIMÍR PALKO

В работе определено понятие доминированости и сингулярности обобщенных состояний на логике и доказана следующая теорема: Если ω — конечное неотрицательное обобщенное состояние и m — конечное обобщенное состояние на логике замкнутых подпространств сепарабельного пространства Гильберта, то m является суммой двух обобщенных состояний m_1 и m_2, где m_1 доминировано ω и m_2 сингулярно на ω. Показано также, что неотрицательность m не гарантирует неотрицательность m_1 и m_2.

Author's address: 812 19 Bratislava, Mlynská dolina (Katedra matematiky Elektrotechnickej fakulty SVŠT).