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SOME NEW RESULTS ABOUT THE SHORTNESS EXPONENT IN POLYHEDRAL GRAPHS

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Summary. The shortness exponent $\sigma(\Gamma)$ of a family $\Gamma$ of graphs $G$ is defined as $\sigma(\Gamma) := \liminf_{G \in \Gamma} (\log h(G))/(\log n(G))$, where $n(G)$ or $h(G)$ denotes the number of vertices of $G$ or the maximum number of vertices of $G$ belonging to a circuit, respectively.

The paper deals with shortness exponents of families of regular graphs of polyhedra having the smallest number of types of faces and a shortness exponent $< 1$.

Keywords: Shortness exponent, polyhedral graphs.

We denote by $n(G)$ the number of vertices and by $h(G)$ the number of vertices in a maximum circuit of a graph $G$. In the following, we deal with families $\Gamma$ of graphs containing non-Hamiltonian members $G$, that means $h(G) < n(G)$ and, moreover, for any $\varepsilon > 0$ there exists a $G \in \Gamma$, such that $h(G)/n(G) < \varepsilon$.

A suitable concept for estimating the length of a maximum circuit is the shortness exponent $\sigma(\Gamma)$ of a family $\Gamma$ of graphs $G$:

$$\sigma(\Gamma) := \liminf_{G \in \Gamma} \frac{\log h(G)}{\log n(G)}.$$

In this paper, we only study families of polyhedral graphs. Let us denote by $\Gamma_r$, $r \in \{3, 4, 5\}$ the family of all regular polyhedral graphs of degree $r$. As is well-known, a graph $G$ is polyhedral iff $G$ is planar and three-connected. Let us denote by $\Gamma_r(p_1, \ldots, p_m)$ the subfamily of $\Gamma_r$ containing at most $m$ types of faces, namely, $p_1$-gons, $p_2$-gons, ..., $p_m$-gons.

J. Zaks [4] searched for the minimum $m(r)$ such that there exist $m = m(r)$ integers $p_1, p_2, \ldots, p_m$ with the property that $\sigma(\Gamma_r(p_1, p_2, \ldots, p_m)) < 1$. P. J. Owens [3] proved that for $r \in \{4, 5\}$ the inequality $m(r) \leq 3$ holds. We can prove

Theorem 1. $m(r) = 2$ for $r \in \{3, 4, 5\}$.

If $\Gamma_r(p(r), q(r)) \neq \emptyset$ and $p'(r) < q(r)$ then $p(3) \in \{3, 4, 5\}$, $p(4) = p(5) = 3$ (see [2]).

We can prove

Theorem 2.

$$\sigma(\Gamma_3(4, K)) \leq \frac{\log 44}{\log 45} \quad \text{for any odd} \quad K \geq 21,$$
In this paper we will only prove the first inequality. (The proofs of the remaining estimates of the shortness exponents indicated in Theorem 2 can be seen from [2].) To this aim we construct a sequence \( \{G_i\} \) of 3-regular polyhedral graphs containing only 4-gons and \( K \)-gons and satisfying

\[
\lim_{i \to \infty} \frac{\log h(G_i)}{\log n(G_i)} \leq \frac{\log 44}{\log 45}.
\]

Let \( G_0 \) be the graph of a cube with exactly one distinguished (black) vertex.

\( G_{i+1} \) arises from \( G_i \) by replacing each of the black vertices by a suitable figure \( Z \) still to be constructed.

**Definition.** Let \( i, j, m \) be integers and \( G \) be a graph with following properties:

(i) \( G \) is planar, 3-connected and 3-regular.
(ii) \( G \) contains a vertex \( P \) incident with an \((i+1)\)-gon, a \((j+1)\)-gon and an \((m+1)\)-gon. All the other faces of \( G \) are 4-gons or \( K \)-gons.
(iii) \( G \) contains a vertex \( Q \) \((Q \neq P)\) incident with three 4-gons. Exactly one of these vertices will be distinguished. We call it black.

A **figure** \((i, j, m)\) is the object arising from \( G \) by splitting up the vertex \( P \) into 3 half-edges (in Fig. 1 \( G \) is the graph of a cube and we obtain by splitting up an arbitrary vertex a figure \((3, 3, 3)\)).

![Figure 1](image-url)

Let us now construct the figure \( Z \). This construction proceeds in two steps:

**Step 1:** We start with the figure \( E \) arising from the well-known non-Hamiltonian
Grinberg graph (see [5]) splitting off some vertex (remark: $E$ has the property that a path through $E$ which connects any two arbitrary halfedges leaves out at least one vertex).

**Step 2:** Each of the vertices of $E$ is to be replaced by a suitable figure $(i, j, m)$ and we obtain the figure $Z$ shown in Fig. 2. As one can easily see, we need eleven different figures $(i, j, m)$, namely $(3, 3, 3), (2, 3, 4), (2, K - 16, K - 17), (2, 3, 9), (2, 3, 8), (2, 3, 7), (2, 3, 10), (8, 9, K - 17), (2, 3, 5), (2, K - 12, K - 13), (2, K - 13, K - 14).

Fig. 3 shows a figure $(3, 3, 3)$.

For constructing the eleven figures $(i, j, m)$ needed let us introduce some operations taking into consideration $2 \leq i, j, m \leq K - 2$ (We use the abbreviation: $(i, j, m)$ $A(u, v, w)$, if we obtain a figure $(u, v, w)$ by applying the operation $A$ to the figure $(i, j, m)$):

$a$: $(i, j, m) a(K - i, j + 2, m + 2)$, see Fig. 4 for $K = 13$. 

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Bold-face italics (e.g. \(i, K - i\)) means that the operation is performed at the exterior face the border of which contains \(i\) vertices.

\[
\beta: \ (i, j, m) \beta(i, j + 4, m + 4) := (i, j, m) \alpha(K - i, j + 2, m + 2)
\alpha(i, j + 4, m + 4).
\]

\[
\gamma: \ (i, j, m) \gamma(i + 8, j + 8, m + 8) := (i, j, m) \beta(i, j + 4, m + 4)
\beta(i + 4, j + 4, m + 8) \beta(i + 8, j + 8, m + 8).
\]

\[
\delta: \ (K - 2, j, m) \delta(2, j + 1, m + 1), \text{ see Fig. 5 for } K = 13.
\]

\[
\epsilon: \ (2, j, m) \epsilon(2, j + 1, m + 1), \text{ see Fig. 6.}
\]

\[
\varphi: \ (2, 3 + i, 3 + i + j) \varphi(2, 3, j + 3) :=
:= (2, 3 + i, 3 + i + j) \epsilon(2, 3 + i + 1, 3 + i + j + 1)
\epsilon ... \epsilon(2, K - j - 2, K - 2) \delta
(3, K - j - 1, 2) \epsilon(4, K - j, 2)
\epsilon ... \epsilon(j + 2, K - 2, 2) \delta
j + 3, 2, 3).
\]
\[ \psi: (2, 3, j + 11) \psi(2, 3, j + 3) := (2, 3, j + 11) \beta(6, 7, j + 11) \]
\[ \alpha(K - 6, 9, j + 13) \beta(K - 2, 13, j + 13) \]
\[ \delta(2, 14, j + 14) \varphi_{11}(2, 3, j + 3). \]

Given a figure \((2, 3, 5)\) we get the other seven figures \((i, j, m)\) in the following way.

Fig. 5,

\[ \begin{align*}
\text{Fig. 6,} \\
\text{Fig. 7,} \\
\end{align*} \]

\(\cong (4, 6, 7)\)

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\((2, 3, 5) \varepsilon(2, 4, 6) \alpha(4, 6, K - 6)\) and according to Fig. 7, \((4, 6, 7) \alpha
\(6, K - 6, 9) \beta(10, K - 2, 9) \delta(11, 2, 10) \varepsilon^{K-13}(K - 2, 2, K - 3) \delta(2, 3, K - 2)
\delta(3, 4, 2).

- \((3, 4, 2) \varepsilon^{K-20}(K - 17, K - 16, 2),\) analogously \((K - 12, K - 13, 2)\) and \((K -
- 13, K - 14, 2),\)

- \((2, 3, 5) \beta(6, 7, 5) \alpha(K - 6, 9, 7) \beta(K - 2, 13, 7) \delta(2, 14, 8) \varphi(2, 9, 3).

- \((2, 3, 5) \beta(6, 7, 5) \alpha(K - 6, 9, 7)\) and according to Fig. 8, \((4, 11, 8) \alpha

- \((2, 3, 8) \alpha(4, 7, K - 10) \varepsilon^2(2, K - 2, 15) \delta(3, 2, 16) \psi(3, 2, 8).

- \((2, 3, 5) \varepsilon^2(2, 5, 10) \alpha(4, 7, K - 10)\) see Fig. 9

- \((8, 6, 8) \alpha(10, K - 6, 10) \beta(10, K - 2, 14) \delta(11, 2, 15) \varphi(3, 2, 7).

- \((2, 3, 4) \varepsilon^2(2, 5, 6) \alpha(4, 7, K - 6) \beta(4, 11, K - 2) \delta(5, 12, 2) \varphi(3, 10, 2).

- \((2, 3, 9) \beta(6, 3, 13) \beta(6, 7, 17) \alpha(8, 9, K - 17).

Now, let us construct a \((2, 3, 5)\). We have to distinguish four cases.

Case 1, \(K = 8N + 7, N \geq 2.

\(\gamma^{N-1}(8N - 5, 8N - 5, 8N - 5) \beta(8N - 5, 8N - 1, 8N - 1)\)
\(\alpha(8N - 3, 8, 8N + 1) \beta(8N + 1, 8, 8N + 5) \delta(8N + 2, 9, 2) \varepsilon^2(8N + 5, 12, 2)
\delta(2, 13, 3) \psi(2, 5, 3).

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Case 2, $K = 8N + 5$, $N \geq 2$.

$$(3, 3, 3) \varphi^{-1}(8N - 5, 8N - 5, 8N - 5) \beta(8N - 5, 8N - 1, 8N - 1)$$

$$\beta(8N - 1, 8N - 1, 8N + 3) \delta(8N, 8N, 2) \epsilon(8N + 1, 8N + 1, 2) \alpha(4, 8N + 3, 4)$$

$$\delta(5, 2, 5) \alpha(7, 4, K - 5)$$
on the one hand according to Fig. 10, $(8, 11, K - 4)$

$$\vartriangle (K-4, 11, 8)$$

Fig. 10,

$$\alpha(10, K - 11, K - 2) \delta(11, K - 10, 2) \beta^2(11, K - 2, 10) \delta(12, 2, 11) \varphi(4, 2, 3);$$
on the other hand, $(7, 4, K - 5) \alpha(9, K - 4, K - 3)$ and according to Fig. 11,

$$\Delta (3, 13, K - 2)$$

Fig. 11,

$$(3, 13, K - 2) \delta(4, 14, 2) \varphi_1(3, 13, 2) \psi(3, 5, 2).$$

Case 3, $K = 8N + 3$, $N \geq 3$.

$$(3, 3, 3) \varphi^{-1}(8N - 5, 8N - 5, 8N - 5) \alpha(8, 8N - 3, 8N - 3)$$

$$\beta(12, 8N - 3, 8N + 1) \delta(13, 8N - 2, 2) \epsilon^2(16, 8N + 1, 2) \delta(17, 2, 3)$$

$$\psi(9, 2, 3) \alpha'(K - 9, 4, 5) \alpha(K - 7, 6, K - 5) \alpha'(K - 5, K - 6, K - 3)$$

$$\alpha(K - 3, K - 4, 3) \alpha(3, K - 2, 5) \delta(4, 2, 6) \varphi(3, 2, 5).$$

Case 4, $K = 8N + 1$, $N \geq 3$.

$$(3, 3, 3) \varphi^{-1}(8N - 5, 8N - 5, 8N - 5) \alpha(6, 8N - 3, 8N - 3) \alpha(8, 4, 8N - 1)$$

$$\delta(2, 9, 5) \epsilon(10, 6, 2) \alpha(12, K - 6, 4)$$
together with $(*)$ according to Fig. 12 yields

$$(5, 10, 14) \alpha(7, K - 10, 16) \beta^2(15, K - 2, 16) \delta(16, 2, 17) \varphi_{13}(3, 2, 4) (**).$$
From (*) we get \( (9, 5, 2) \) \( \varphi_2(7, 3, 2) \alpha(9, K - 3, 4) \) together with (**) according to Fig. 13 yields a \( (5, 7, 10) \) \( \alpha(7, 9, K - 10) \) \( \beta(7, 13, K - 6) \) \( \beta(11, 13, K - 2) \) \( \delta(12, 14, 2) \) \( \varphi_3(3, 5, 2) \).

In all cases we have found a figure \( (2, 3, 5) \), that means for all odd \( K \geq 21 \) we have found the necessary eleven types of \((i, j, m)\) listed in Step 2.

Constructing the figures \((i, j, m)\) needed we have always started with a figure \((3, 3, 3)\), that means, each figure \((i, j, m)\) contains at least one vertex incident with three 4-gons. Hence the figure \( Z \) contains exactly 45 black vertices and an arbitrary path through \( Z \) connecting any two halfedges avoids at least one black vertex. Obviously \( Z \) is a figure \((K - 3, K - 3, K - 3)\).

By induction, if \( G_t \) contains only 4-gons and \( K \)-gons (which is really true for \( l = 0 \)), then after replacing all black vertices of \( G_t \) by a figure \( Z \), the resulting graph \( G_{t+1} \) has only 4-gons and \( K \)-gons as well.

The estimate \( \log 44/\log 45 \) for the shortness exponent may be established in an analogous way as in [1]. Obviously, the number of vertices increases by the factor 45, but the number of vertices in the longest circuit only by the factor 44 if we pass from \( G_t \) to \( G_{t+1} \).
The following problems remain unsolved:
1. What holds in the open cases of Theorem 2, \( K \equiv 0 \pmod{5} \) for \( \Gamma_3(5, K) \) and \( K \equiv 0 \pmod{3} \) for \( \Gamma_4(3, K) \) and \( \Gamma_5(3, K) \)?
2. Conjecture: \( \sigma(\Gamma_3(3, K)) = 1 \) for all \( K \leq 11 \) (obviously, \( \Gamma_3(3, K) = \emptyset \) for \( K \geq 12 \).

References


Souhrn

NOVÉ VÝSLEDKY O ,,SHORTNESS EXPONENTU“ POLYEDRICKÝCH GRAFŮ

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Exponent \( \sigma(\Gamma) \) (,,shortness exponent“) soustavy \( \Gamma \) grafů \( G \) je definován jako \( \sigma(\Gamma) = \liminf (\log h(G)) / (\log n(K)) \), kde \( n(G) \) resp. \( h(G) \) znamená počet vrcholů \( G \) resp. maximální počet vrcholů \( G \), patřících cyklu. V práci se studuje exponent \( G(\Gamma) \) soustav regulárních polyedrických grafů, které mají nejmenší počet typů stěn, a pro něž platí \( \sigma(\Gamma) < 1 \).

Резюме

НОВЫЕ РЕЗУЛЬТАТЫ О ,,ПОКАЗАТЕЛЕ КОРОТКОСТИ“ ПОЛИЭДРИЧЕСКИХ ГРАФОВ

ЙОХЕН ГАРАНТ, ХАНСЙОАХИМ ВАЛТЕР

Показатель \( \sigma(\Gamma) \) (,,показатель короткости“) системы \( \Gamma \) графов \( G \) определяется формулой
\[
\sigma(\Gamma) = \liminf_{G \in \Gamma} (\log h(G)) / (\log n(\Gamma)),
\]
где \( n(G) \) и \( h(G) \) обозначают соответственно число вершин графа \( G \) и максимальное число вершин графа \( G \), принадлежащих циклу. В работе изучается показатель \( G(\Gamma) \) систем регулярных полиэдрических графов, имеющих наименьшее число типов граней и удовлетворяющих \( \sigma(\Gamma) < 1 \).

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