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NOTE ON CONTRACTIVITY OF THE NEUMANN OPERATOR

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Summary. The paper deals with the Neumann operator on the space of signed measures on boundary, for which the measure of the whole space equals zero. An example of a set, the complement of which is a non-convex bounded set, and at the same time the corresponding Neumann operator is contractive on this space, is shown.

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It was shown in [1] that the Neumann problem for the Laplace equation on an open set $G \subset R^m$ $(m \geqq 2)$ with compact boundary $\partial G = B$ can be investigated without apriori smoothness restrictions on B. The corresponding generalized problem leads to the operator equation

$$(I + U)\, \nu = 2\mu \tag{1}$$

on $\mathscr{C}'(B)$, the linear space of all finite signed measures carried by B. Denote $\mathscr{C}'_0(B) := \mathscr{C}'(B) \cap \{\nu;\, \nu(B) = 0\}$. Both spaces are Banach spaces with respect to the norm defined as the total variation.

Let us recall some facts: necessary and sufficient conditions to have $(I + U)\, \nu \in \mathscr{C}'(B)$ for every $\nu \in \mathscr{C}'(B)$ were obtained by Král (see [1], Theorem 1.13). If G is unbounded, then $\|U\| \leqq 1$ if and ond if $C := R^m - G$ is convex. If C is convex then $\|U\| = 1$ and, moreover, necessary and sufficient conditions for contractivity of U on the space $\mathscr{C}'_0(B)$ (which go back to C. Neumann) are known; see [1], Theorem 3.1, Theorem 3.5. Thus the following two problems are of interest: Under which conditions on G is it true that $(I + U)\, \nu \in \mathscr{C}'(B)$ for each $\nu \in \mathscr{C}'_0(B)$? Is the convexity of C also a necessary condition for U to be a contractive operator on $\mathscr{C}'_0(B)$ provided G is unbounded?

In our text we use the same notation as in [1], but for the sake of convenience we recall some definitions.

Let us examine the Neumann problem for the Laplace equation on an open set $G \subset R^m$ $(m \geqq 2)$ with a compact boundary B. We define the generalized normal derivative of a function h harmonic in G as the distribution

$$\langle \varphi, N^G h \rangle = \int_G \operatorname{grad} \varphi(x) \operatorname{grad} h(x)\, dx\,, \quad \varphi \in \mathscr{D}$$

provided $|\text{grad } h|$ is integrable on every bounded open subset of G; as usual, $\mathcal{D} = \mathcal{D}(R^m)$ stands for the class of all infinitely differentiable functions with compact support in R^m. Then $N^G h$ is a distribution with a support in B (see [1], Remark 1.2). Given a finite signed measure μ with a support in B we seek for a function h harmonic in G, such that $N^G h = \mu$. Suppose for a moment that G is bounded by a smooth closed surface B with the area element ds and the exterior normal $n = (n_1, \ldots, n_m)$, and that the partial derivatives with respect to the i-th variable $\partial_i h$ $(i = 1, \ldots, m)$ extend from G to continuous functions on the whole $G \cup B$; the Gauss-Green formula yields

$$\langle \varphi, N^G h \rangle = \int_B \varphi\left(\sum_{i=1}^m n_i\, \partial_i h\right) \mathrm{d}s\,, \quad \varphi \in \mathcal{D}\,.$$

Consequently, $N^G h$ is a natural weak characterization of the normal derivative $\sum n_i\, \partial_i h = \partial h / \partial n$ and the above problem is a generalization of the classical Neumann problem.

We shall try to find the solution $h(x)$ of the Neumann problem in the form of a potential

$$\mathcal{U}\, v(x) = \int_{R^m} h_z(x)\, \mathrm{d}v(z)\,,$$

where

$$h_z(x) = \frac{1}{(m-2)\,A}\, |x - z|^{2-m} \quad \text{if} \quad m > 2\,,$$

$$\frac{1}{A} \log \frac{1}{|x - z|} \quad \text{if} \quad m = 2\,;$$

here $A \equiv A_m = 2\pi^{m/2}/\Gamma(m/2)$ is the area of the unit sphere in R^m and v denotes a finite signed measure with support in B. Since $\mathcal{U}v$ is a harmonic function on $R^m - B$, our problem reduces to finding such $v \in \mathscr{C}'(B)$ that

$$N^G \mathcal{U} v = \mu\,.$$

We denote by $\Omega_r(y)$ the ball with radius r and centre y, and by $\varkappa_k$ the k-dimensional Hausdorff measure. It suffices to investigate our problem only for those sets G for which the boundary B coincides with $\partial_e G = \{y \in R^m;\ \forall r > 0\colon \varkappa_m(\Omega_r(y) \cap G) > 0,\ \varkappa_m(\Omega_r(y) - G) > 0\}$, the essential boundary of G (see [1], Remark 1.14). In what follows we assume that $B = \partial_e G$.

For the investigation of $N^G \mathcal{U}$ we shall introduce several useful concepts. A point $y \in S \subset R^m$ will be termed a hit of S on G if for every $r > 0$ both $\varkappa_1(\Omega_r(y) \cap S \cap G) > 0$ and $\varkappa_1(\Omega_r(y) \cap (S \setminus G)) > 0$ hold. (In our applications S will usually be a straight line or a half-line.) Put

$$v^G(y) = \sup\left\{\int_G \text{grad}\, \psi(x) \,.\, \text{grad}\, h_y(x)\, \mathrm{d}x\,;\quad \psi \in \mathcal{D}\,,\quad |\psi| \leqq 1\,,\right.$$

$$\operatorname{spt} \psi \subset R^m - \{y\}\Big\} ,$$

where spt ψ denotes the support of ψ and grad $\psi = (\partial_1 \psi, \ldots, \partial_m \psi)$ is the gradient of ψ. According to [1], Corollary 1.11,

$$v^G(y) = \frac{1}{A} \int_\Gamma n^G(\theta, y) \, \mathrm{d}\varkappa_{m-1}(\theta) ,$$

where Γ is the unit sphere and $n^G(\theta, y)$ is the number (possibly 0 or $+\infty$) of all hits of $\{y + t\theta; t > 0\} = P(y, \theta)$ on G, for each $\theta \in \Gamma$.

Theorem 1. *$N^G \mathcal{U}$ maps $\mathcal{C}'(B)$ into $\mathcal{C}'(B)$ if and only if*

$$V^G = \sup_{y \in B} v^G(y) < \infty .$$

Proof. See [1], Theorem 1.13.

Now we prove a similar theorem for the space $\mathcal{C}'_0(B)$. We define the perimeter of the set G by

$$P(G) = \sup_w \int_G \operatorname{div} w(x) \, \mathrm{d}x ,$$

where $w = (w_1, \ldots, w_m)$ runs over all vector-valued functions with components $w_j \in \mathcal{D}$ such that $|w|^2 = \sum w_j^2 \leqq 1$, and where $\operatorname{div} w(x) = \sum \partial_j w_j(x)$ is the divergence of w. Denote

$$P_i(G) = \sup \left\{ \int_G \partial_i \psi(x) \, \mathrm{d}x; \ \psi \in \mathcal{D}, \ |\psi| \leqq 1 \right\} \quad \text{for} \quad i = 1, \ldots, m .$$

Lemma 1.

$$\sup_{i=1,\ldots,m} P_i(G) \leqq P(G) \leqq \sum_{i=1}^m P_i(G)$$

and

$$P_i(G) = \int_{R^{m-1}} p_i^G(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m) \, \mathrm{d}y_1 \ldots \mathrm{d}y_{i-1} \, \mathrm{d}y_{i+1} \ldots \mathrm{d}y_m ,$$

where $p_i^G(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m)$ is the number (possibly 0 or $+\infty$) of all hits of $\{(y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_m); t \in R\}$ on G.

Proof. See [2], Definition 2, Definition 15, Lemma 11 and Lemma 17.

Lemma 2. *If $N^G \mathcal{U} v \in \mathcal{C}'(B)$ for each $v \in \mathcal{C}'_0(B)$ then $P(G) < \infty$.*

Proof. We distinguish two cases:

1) For every $x \in B$ there are $z^1, \ldots, z^{m+1} \in B - \{x\}$ such that the matrix

$$(*) \qquad \left(\frac{z^1 - x}{|z^1 - x|^m} - \frac{z^{m+1} - x}{|z^{m+1} - x|^m}, \ldots, \frac{z^m - x}{|z^m - x|^m} - \frac{z^{m+1} - x}{|z^{m+1} - x|^m}\right)$$

is regular.

2) There is an $x \in B$ such that the matrix $(*)$ is singular for every $z^1, \ldots, z^{m+1} \in B - \{x\}$.

Let us consider the case 1. According to Lemma 1 it suffices to prove that $P_j(G) < \infty$ for $j = 1, \ldots, m$. Let, for example, $j = 1$. If $x \in B$ then there are $z^1, \ldots, z^{m+1} \in B - \{x\}$ such that the determinant $D(z^1, \ldots, z^{m+1}, x)$ of $(*)$ is not 0. Since the determinant $D(z^1, \ldots, z^{m+1}, \cdot)$ is a continuous function on $R^m - \{z^1, \ldots, z^{m+1}\}$, there exists $r(x) > 0$ such that $\Omega_{r(x)}(x) \cap \{z^1, \ldots, z^{m+1}\} = \emptyset$ and $D(z^1, \ldots, z^{m+1}, y) \neq 0$ for each $y \in \Omega_{r(x)}(x)$. Thus the vectors

$$(z^1 - y)/|z^1 - y|^m - (z^{m+1} - y)/|z^{m+1} - y|^m, \ldots, (z^m - y)/|z^m - y|^m - (z^{m+1} - y)/|z^{m+1} - y|^m$$

are linearly independent for all $y \in \Omega_{r(x)}(x)$; such a ball can be found for every $x \in B$. Since B is compact and $B \subset \bigcup\{\Omega_{r(x)}(x);\ x \in B\}$, there are $x^1, \ldots, x^n \in B$ such that

$$B \subset \bigcup_{i=1}^{n} \Omega_{r(i)}(x^i),$$

where $r(i) = r(x^i)$. There are $\alpha_0, \ldots \alpha_n \in C^\infty(R^m)$ such that $0 \leqq \alpha_i \leqq 1$, $\sum \alpha_i = 1$ on $\bar{G}$, spt $\alpha_0 \subset G$ and spt $\alpha_i \subset \Omega_{r(i)}(x^i)$ for $i = 1, \ldots, n$. Since

$$\int_G \partial_1 \psi(x)\, dx = \sum_{i=0}^{n} \int_G \alpha_i(x)\, \partial_i \psi(x)\, dx$$

for any $\psi \in \mathcal{D}$ it suffices to prove that

$$\sup\left\{\int_G \alpha_i(x)\, \partial_1 \psi(x);\ \psi \in \mathcal{D},\ |\psi| \leqq 1\right\} < \infty$$

for $i = 0, \ldots, n$. Since $\partial_1 \alpha_0 \in \mathcal{D}$ we have for $i = 0$

$$\left|\int_G \alpha_0(x)\, \partial_1 \psi(x)\, dx\right| = \left|\int_{R^m} \alpha_0(x)\, \partial_1 \psi(x)\, dx\right| = \left|-\int_{R^m} \psi(x)\, \partial_1 \alpha_0(x)\, dx\right| \leqq$$
$$\leqq \int_{R^m} |\partial_1 \alpha_0(x)|\, dx < \infty\,.$$

Now let $i \in \{1, \ldots, n\}$, for example $i = 1$. There are $z^1, \ldots, z^{m+1} \in B$ such that $(z^1 - x)/|z^1 - x|^m - (z^{m+1} - x)/|z^{m+1} - x|^m, \ldots, (z^m - x)/|z^m - x|^m - (z^{m+1} - x)/|z^{m+1} - x|^m$ are linearly independent vectors for all $x \in$ spt α_1. Therefore, there are $a_k(x) \in C^\infty(\Omega_{r(1)}(x^1))$, the space of all infinitely differentiable functions on $\Omega_{r(1)}(x^1)$, such that

$$(1, 0, 0, \ldots, 0) = \sum_{k=1}^{m} a_k(x)\left((z^k - x)/|z^k - x|^m - (z^{m+1} - x)/|z^{m+1} - x|^m\right).$$

Denote by δ_x the Dirac measure with the support $\{x\}$. If $\psi \in \mathscr{D}$, $|\psi| \leqq 1$ then we have

$$\left| \int_G \alpha_1(x)\, \partial_1 \psi(x)\, dx \right| =$$

$$= \left| \int_G \alpha_1(x) \sum_{k=1}^m ((z^k - x)/|z^k - x|^m - (z^{m+1} - x)/|z^{m+1} - x|^m)\, a_k(x) \,.\, \operatorname{grad} \psi(x)\, dx \right| \leqq$$

$$\leqq \sum_{k=1}^m \left| \int_G ((z^k - x)/|z^k - x|^m - (z^{m+1} - x)/|z^{m+1} - x|^m) \,.\, \operatorname{grad} (\alpha_1(x)\, a_k(x)\, \psi(x))\, dx - \right.$$

$$- \int_G ((z^k - x)/|z^k - x|^m - (z^{m+1} - x)/|z^{m+1} - x|^m) \,.$$

$$\left. .\, \psi(x) \operatorname{grad} (\alpha_1(x)\, a_k(x))\, dx \right| \leqq$$

$$\leqq \sum_{k=1}^m |\langle N^G \mathscr{U}(\delta_{z^k} - \delta_{z^{m+1}}),\ \alpha_1(x)\, a_k(x)\, \psi(x) \rangle| +$$

$$+ \sum_{k=1}^m \int_G |(z^k - x)/|z^k - x|^m - (z^{m+1} - x)/|z^{m+1} - x|^m|\, |\operatorname{grad} (\alpha_1(x)\, a_k(x))|\, dx \leqq$$

$$\leqq \sum_{k=1}^m \| N^G \mathscr{U}(\delta_{z^k} - \delta_{z^{m+1}}) \| \sup_{x \in \operatorname{spt} \alpha_1} |a_k(x)| +$$

$$+ \sum_{k=1}^m \int_{\operatorname{spt} \alpha_1} (|z^k - x|^{1-m} + |z^{m+1} - x|^{1-m})\, |\operatorname{grad} (\alpha_1(x)\, a_k(x))|\, dx < \infty\,.$$

Remark that sice $\delta_{z^k} - \delta_{z^{m+1}} \in \mathscr{C}'_0(B)$ we have $N^G \mathscr{U}(\delta_{z^k} - \delta_{z^{m+1}}) \in \mathscr{C}'(B)$, and that $(|z^k - x|^{1-m} + |z^{m+1} - x|^{1-m})\, |\operatorname{grad} (\alpha_1(x)\, a_k(x))$ is a finite continuous function on spt α_1 and $\alpha_1(x)\, a_k(x)\, \psi(x) \in \mathscr{D}$. Thus the perimeter of G is finite.

Consider now the case 2. There is an $x \in B$ such that the matrix (*) is singular for every $z^1, \ldots, z^{m+1} \in B - \{x\}$. To simplify the situation we may assume that $x = 0$. Thus for all $z \in B - \{0\}$ the points $z/|z|^m$ are located in a single hyperplane L. We may select such a coordinate system preserving the origin that $L = \{y;\, y_1 = t\}$ for some $t \geqq 0$. Thus $z_1 = t|z|^m$ for every $z \in B$ and hence

$$t^2 (\sum_{j=1}^m z_j^2)^m - z_1^2 = 0\,. \tag{2}$$

If $t = 0$ then $B \subset \{z;\, z_1 = 0\}$ and an easy calculation yields $P(G) = 0$. Let $t > 0$. According to Lemma 1 it suffices to prove that $P_i(G) < \infty$ for $i = 1, \ldots, m$. Fix i. Every hit z of $\{(y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_m);\ t \in R\}$ on G lies in B. Such z satisfies (2) and $z_j = y_j$ for $j \neq i$. Since $t^2(\sum z_j^2)^m - z_1^2$ is a nonzero polynomial in the variable z_i the order of which does not exceed $2m$, for fixed $z_1, \ldots, z_{i-1}, \ldots, z_m$ there exist at most $2m$ different solutions of the equation (2). Thus

$p_i^G(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m) \leqq 2m$ for any choice of $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m$. Since B is compact there is $K > 0$ such that $B \subset \Omega_K(0)$. It follows from Lemma 1 that

$$P_i(G) \leqq \int_{\prod_{j=1}^{m-1}\langle -K,K\rangle} p_i^G(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m)\, dy_1 \ldots dy_{i-1}\, dy_{i+1} \ldots dy_m \leqq$$
$$\leqq 2m(2K)^{m-1} < \infty$$

and hence $P(G) < \infty$ again.

Theorem 2. *$N^G\mathcal{U}v \in \mathscr{C}'(B)$ for each $v \in \mathscr{C}_0'(B)$ if and only if $V^G < \infty$.*

Proof. If $V^G < \infty$ then $N^G\mathcal{U}v \in \mathscr{C}'(B)$ for each $v \in \mathscr{C}'(B)$ according to Theorem 1.

Now let us suppose that $N^G\mathcal{U}v \in \mathscr{C}'(B)$ for each $v \in \mathscr{C}_0'(B)$. With any $\varphi \in \mathscr{D}$ we associate the linear functional L_φ on $\mathscr{C}_0'(B)$ defined by

$$\langle v, L_\varphi\rangle = \langle \varphi, N^G\mathcal{U}v\rangle\,, \quad v \in \mathscr{C}_0'(B)\,.$$

If $P_\varphi = G \cap \operatorname{spt} \varphi$ and $c_\varphi = \sup\{|\operatorname{grad} \varphi(x)|;\ x \in R^m\}$, we have

$$\left|\langle v, L_\varphi\rangle\right| = \left|\int_G \int_B \operatorname{grad} \varphi(x) \cdot \operatorname{grad} h_y(x)\, dv(y)\, dx\right| \leqq$$
$$\leqq \int_B \int_G |\operatorname{grad} \varphi(x) \cdot \operatorname{grad} h_y(x)|\, dx\, d|v|(y) \leqq$$
$$\leqq c_\varphi \int_B \int_{P_\varphi} \frac{1}{A} |x - y|^{1-m}\, dx\, d|v|(y) \leqq c_\varphi \operatorname{diam}(P_\varphi \cup B)\, \|v\|\,.$$

Thus L_φ is a bounded linear functional on the Banach space $\mathscr{C}_0'(B)$. If $v \in \mathscr{C}_0'(B)$ then $N^G\mathcal{U}v \in \mathscr{C}'(B)$. But $N^G\mathcal{U}v \in \mathscr{C}'(B)$ if and only if $\sup\{\langle \varphi, N^G\mathcal{U}v\rangle;\ \varphi \in \mathscr{D},\ |\varphi| \leqq 1\} < \infty$. Hence

$$\sup\{\langle v, L_\varphi\rangle;\ \varphi \in \mathscr{D},\ |\varphi| \leqq 1\} = \sup\{\langle \varphi, N^G\mathcal{U}v\rangle;\ \varphi \in \mathscr{D},\ |\varphi| \leqq 1\} < \infty\,.$$

Applying the uniform boundedness principle we obtain

$$\sup_{\varphi \in \mathscr{D}, |\varphi| \leqq 1} \|L_\varphi\| < \infty\,.$$

Thus there exists an M such that $0 < M < \infty$ and

$$|\langle \varphi, N^G\mathcal{U}v\rangle| \leqq M\|v\| \tag{3}$$

for each $\varphi \in \mathscr{D}$, $|\varphi| \leqq 1$, $v \in \mathscr{C}_0'(B)$.

Now let $x \in B$. Choose $y \in B - \{x\}$, $0 < \delta < |x - y|/2$. Since $P(G) < \infty$, $P(G - \Omega_\delta(y)) < \infty$ is valid. According to [1], Proposition 2.11 we have

$$v^H(y) \leqq \frac{1}{A} P(H)\, \delta^{1-m} < \infty\,,$$

where $H = G - \Omega_\delta(y)$. If $\psi \in \mathscr{D}$, $|\psi| \leqq 1$, spt $\psi \subset \Omega_\delta(x)$ then

$$\left|\int_G \operatorname{grad} \psi(z) \,.\, \operatorname{grad} h_x(z)\, dz\right| \leqq$$

$$\leqq \left|\int_G \operatorname{grad} \psi(z) \,.\, (\operatorname{grad} h_x(z) - \operatorname{grad} h_y(z))\, dz\right| +$$

$$+ \left|\int_G \operatorname{grad} \psi(z) \,.\, \operatorname{grad} h_y(z)\, dz\right| \leqq 2M + \left|\int_H \operatorname{grad} \psi(z) \,.\, \operatorname{grad} h_y(z)\, dz\right| \leqq$$

$$\leqq 2M + v^H(y)\,.$$

Therefore, for every $x \in B$ there are $r(x) > 0$ and $0 < K(x) < \infty$ such that

$$\left|\int_G \operatorname{grad} \psi(z) \,.\, \operatorname{grad} h_x(z)\, dz\right| \leqq K(x)$$

for each $\psi \in \mathscr{D}$, $|\psi| \leqq 1$, spt $\psi \subset \Omega_{r(x)}(x)$. Since $\bigcup_{x \in B} \Omega_{r(x)}(x) \supset B$ and B is a compact set, there are $x^1, \ldots, x^n \in B$ such that $\bigcup_{i=1}^{n} \Omega_{r(i)}(x^i) \supset B$, where $r(i) = r(x^i)$. Further, for $i = 1, \ldots, n$ there exist $\alpha_i \in \mathscr{D}$, $0 \leqq \alpha_i \leqq 1$, spt $\alpha_i \subset \Omega_{r(i)}(x^i)$ such that $\alpha = \sum \alpha_i$ coincides with 1 on the neighbourhood of B.

Let $x \in B$, $\psi \in \mathscr{D}$, $|\psi| \leqq 1$. Since $\psi \,.\, \alpha = \psi$ on the neighbourhood of B, it is true, according to [1], Remark 1.2, that

$$\left|\int_G \operatorname{grad} \psi(z) \,.\, \operatorname{grad} h_x(z)\, dz\right| =$$

$$= \left|\int_G \operatorname{grad} (\psi(z) \,.\, \alpha(z)) \,.\, \operatorname{grad} h_x(z)\, dz\right| \leqq$$

$$\leqq \sum_{i=1}^{n} \left|\int_G \operatorname{grad} (\alpha_i(z)\, \psi(z)) \,.\, \operatorname{grad} h_x(z)\, dz\right| \leqq$$

$$\leqq \sum_{i=1}^{n} \left|\int_G \operatorname{grad} (\alpha_i(z)\, \psi(z)) \,.\, (\operatorname{grad} h_x(z) - \operatorname{grad} h_{x^i}(z))\, dz\right| +$$

$$+ \sum_{i=1}^{n} \left|\int_G \operatorname{grad} (\alpha_i(z)\, \psi(z)) \,.\, \operatorname{grad} h_{x^i}(z)\, dz\right| \leqq$$

$$\leqq 2Mn + \sum_{i=1}^{n} K(x^i)\,.$$

Hence $v^G(x) \leqq 2Mn + \sum K(x^i)$ and $V^G = 2Mn + \sum K(x^i) < \infty$.

It is possible to verify that if $V^G < \infty$ then $N^G \mathscr{U} v \in \mathscr{C}_0'(B)$ for each $v \in \mathscr{C}_0'(B)$. (See [1], Proposition 2.8 and Proposition 2.20.) Now we shall investigate $N^G \mathscr{U}$ as an operator on the space $\mathscr{C}'(B)$ under the assumption $V^G < \infty$. If we suppose $V^G < \infty$ then the operator $N^G \mathscr{U}$ is even a bounded linear operator on $\mathscr{C}'(B)$. But the investigation of the operator $U = 2N^G\mathscr{U} - I$ is more convenient than the investigation of the operator $N^G\mathscr{U}$. Thus our problem is reduced to the study of the equation (1). For every $\nu \in \mathscr{C}'(B)$ and for every $f \in \mathscr{C}(B)$ we have

$$\langle f, U\nu\rangle = \int_B \int_B f(z)\, d\tau_x(z)\, d\nu(x),$$

where τ_x is for every $x \in B$ a finite signed measure on B and $d\tau_x(z) = [2d_G(x) - 1] \, . \, d\delta_x(z) - 2n^G(z) \, . \operatorname{grad} h_x(z)\, d\varkappa_{m-1}(z)$ (see [1], pp. 72, 73 and Proposition 2.20). We denote by

$$d_G(z) = \lim_{r \to 0+} \varkappa_m(\Omega_r(z) \cap G)/\varkappa_m(\Omega_r(z))$$

the density of G at z and by $n^G(z)$ Federer's normal of G at z. A vector $n^G(z) \in \Gamma$ is termed Federer's normal of G at $z \in R^m$, if the symmetric difference of G and the half-space $\{x \in R^m;\ (x - z) \, . \, n^G(z) > 0\}$ has the m-dimensional density equal to zero at z; otherwise we put $n^G(z) = 0$. If there is such a vector $n^G(z)$, then it is unique and thus the ordinary interior normal of G at z and Federer's normal of G at z coincide, provided the former exists. The measure τ_x is supported by $\hat{B} = \{z \in R^m;\ |n^G(z)| > 0\}$, the reduced boundary of G.

The operator U is a dual operator to the Neumann operator of the arithmetical mean (see [1], Proposition 2.20, and the notation on p. 72) which acts on $\mathscr{C}(B)$, the space of continuous functions on B with the usual sup norm. But the Neumann operator can be defined as an operator on $\mathscr{C}(B)$ not only for any open G but even for any Borel set G with compact boundary B provided $\partial B = \partial_e B$ and $V^G < \infty$ (V^G is defined in the same manner as in the case of an open G) (see [1], Chapter 2). Moreover, $T^G = -T^C$, where T^G is the Neumann operator corresponding to G and C is the complement of G (see [1], p. 73). Thus we may examine the Neumann operator corresponding to C instead of that corresponding to G, and hence to restrict ourselves e.g. to unbounded sets.

If the operator U were a contractive operator on $\mathscr{C}'(B)$ the solution of (1) would have a form $\nu = 2\sum(-1)^k U^k\mu$. If G is unbounded then $\|U\| \leqq 1$ if and only if C is convex (see [1], Theorem 3.1). If C is convex then $\|U\| = 1$ (see [1], Remark 3.2), so that U cannot be contractive on $\mathscr{C}'(B)$; fortunately, there is just one $\varrho \in \mathscr{C}'(B)$ such that $U\varrho = \varrho$ and $\varrho(B) = 1$. If U is contractive as an operator on $\mathscr{C}'_0(B)$ then we can find the solution of the equation (1) in the form

$$\nu = \mu(B)\,\varrho + 2\sum_{k=0}^{\infty}(-1)^k\, U^k(\mu - \mu(B)\,\varrho)$$

for each $\mu \in \mathscr{C}'(B)$. Indeed, since $(\mu - \mu(B)\,\varrho) \in \mathscr{C}'_0(B)$ and U is contractive on $\mathscr{C}'_0(B)$ we have $\sum\|U^k(\mu - \mu(B)\,\varrho)\| \leqq \sum\|U\|_0^k\, \|\mu - \mu(B)\,\varrho\| < \infty$, where $\|U\|_0$ is the norm of U on $\mathscr{C}'_0(B)$, and the series $\sum(-1)^k\, U^k(\mu - \mu(B)\,\varrho)$ converges in $\mathscr{C}'(B)$. Further, we have

$$(I + U)\, \nu = (I + U)\, \mu(B)\, \varrho + 2(I + U) \sum_{k=0}^{\infty} (-1)^k\, U^k(\mu - \mu(B)\, \varrho) =$$

$$= 2\, \mu(B)\, \varrho + 2 \sum_{k=0}^{\infty} (-1)^k\, U^k(\mu - \mu(B)\, \varrho) +$$

$$+ 2 \sum_{k=0}^{\infty} (-1)^k\, U^{k+1}(\mu - \mu(B)\, \varrho) = 2\, \mu(B)\, \varrho + 2(\mu - \mu(B)\, \varrho) = 2\mu$$

and thus ν is the solution of the equation (1). Denote by $Q_x(C)$ the smallest closed cone with vertex x containing C. If C is convex then U is contractive on $\mathscr{C}'_0(B)$ if and only if $Q_x(C) \cap Q_y(C) \neq C$ for every couple of points $x, y \in B$ (see [1], Theorem 3.5).

Example 1. Let us consider $G = R^2 - \mathrm{cl}\, \Omega_1(0)$. Since $\mathrm{cl}\, \Omega_1(0)$ is a convex set, we have $\|U\| = 1$. Now we are going to examine U on $\mathscr{C}'_0(B)$. If $x, z \in B$, $x \neq z$ we denote

$$\varrho(x, z) = n^G(z) \,.\, \mathrm{grad}\, h_x(z)\,. \tag{4}$$

Since $d_G(x) = \frac{1}{2}$ and $n^G(z) = z$ we have

$$\varrho(x, z) = z \,.\, \frac{1}{A} \frac{x - z}{|x - z|^2} = \frac{1}{A} \frac{-1 + z \,.\, x}{|x - z|^2} =$$

$$= \frac{1}{2A} \frac{-|x|^2 + 2z \,.\, x - |z|^2}{|x - z|^2} = \frac{-1}{2A}\,.$$

If $f \in \mathscr{C}(B)$, $\nu \in \mathscr{C}'_0(B)$ then

$$\langle f, U\nu \rangle = \int_B \int_B f(z)\, \mathrm{d}\tau_x(z)\, \mathrm{d}\nu(x) =$$

$$= -\int_B \int_B 2\varrho(x, z) f(z)\, \mathrm{d}\varkappa_{m-1}(z)\, \mathrm{d}\nu(x) =$$

$$= \int_B \int_B \frac{1}{A} f(z)\, \mathrm{d}\varkappa_{m-1}(z)\, \mathrm{d}\nu(x) = 0\,.$$

As a consequence we can easily derive that $\|U\|_0 = 0 < 1 = \|U\|$.

We need C convex only for $\|U\| \leqq 1$. We know, however, that it may happen that $\|U\|_0 < \|U\|$. In fact, inequality $\|U\|_0 < \|U\| = 1$ holds for all convex C with the only exception described above, and for all such cases U is contractive. In the second part of this note we shall present an example showing that the condition "C is convex" is not necessary for the contractivity of U on $\mathscr{C}'_0(B)$.

In R^2 consider $C = \mathrm{cl}\, \Omega_1(0) \cup \mathrm{cl}\, \Omega_1(a)$, where $0 < a < 1/18$ is fixed. It follows from [1], p. 77 that U is contractive on $\mathscr{C}'_0(B)$ if and only if there is a $q < 1$ such that

$$\|\tau_x - \tau_y\| \leqq 2q \tag{5}$$

for each $x, y \in B$. For every $x, y \in B$ we shall prove the inequalities

$$(6) \qquad \|\tau_x - \tau_y\| \leqq \|\tau_x\| + \|\tau_y\| - 2/3\,,$$

$$(7) \qquad \|\tau_x\| \leqq 1 + 1/4$$

from which we shall obtain

$$\|\tau_x - \tau_y\| \leqq 2\,.\,11/12$$

and eventually, with the help of (5), the required contractivity of U on $\mathscr{C}_0'(B)$.

For every $x, y \in B$, $x \neq y$ (cf. notation (4)) we have

$$\|\tau_x - \tau_y\| = |2\,d_G(x) - 1| + |2\,d_G(y) - 1| + \int_{\hat{B}} 2|\varrho(x, z) - \varrho(y, z)|\,\mathrm{d}\varkappa_1(z)\,.$$

If $z = (z_1, z_2)$ does not coincide with x, y and $z_1 < 0$ or $z_1 > a$ then x, y belong to the same halfplane determined by the tangent line to B at z and $\varrho(x, z)$, $\varrho(y, z)$ are of the same sign. Hence for $x, y \in B$, $x \neq y$ we have

$$\|\tau_x - \tau_y\| \leqq |2\,d_G(x) - 1| + |2\,d_G(y) - 1| +$$

$$+ 2\int_{\{z\in B; z_1 \notin \langle 0, a\rangle\}} [\max(|\varrho(x, z)|, |\varrho(y, z)|) - \min(|\varrho(x, z)|, |\varrho(y, z)|)]\,\mathrm{d}\varkappa_1(z) +$$

$$+ 2\int_{\{z\in B; z_1 \in (0, a)\}} |\varrho(x, z) - \varrho(y, z)|\,\mathrm{d}\varkappa_1(z)\,.$$

The estimate of the integrand in the second integral from above by $|\varrho(x, z)| + |\varrho(y, z)|$ and the relation $|\varrho(x, z)| + |\varrho(y, z)| = \max(|\varrho(x, z)|, |\varrho(y, z)|) + \min(|\varrho(x, z)|, |\varrho(y, z)|)$ give after an easy calculation the inequality

$$(8) \quad \|\tau_x - \tau_y\| \leqq \|\tau_x\| + \|\tau_y\| - 4\int_{\{z\in B; z_1 \notin \langle 0, a\rangle\}} \min(|\varrho(x, z)|, |\varrho(y, z)|)\,\mathrm{d}\varkappa_1(z)\,.$$

To obtain (6) we need a lower estimate for the integral in (8). Since for every $x, z \in B$, $z_1 < 0$

$$2|\varrho(x, z)| = \frac{1}{2\pi}|2z\,.\,(z - x)/|z - x|^2| = \frac{1}{2\pi}\left|1 - \frac{|x|^2 - 1}{|z - x|^2}\right|$$

one gets for $x \in \partial\Omega_1(0)$ immediately $2|\varrho(x, z)| = 1/(2\pi)$; for $x \in B - \partial\Omega_1(0)$, $|z_2| \leqq \sqrt{3}/2$ (see the figure) with the help of $|x - z| \geqq \sin \pi/6 = 1/2$ we obtain $(|x|^2 - 1)/|z - x|^2 \leqq 1/2$ and thus

$$(9) \qquad 2|\varrho(x, z)| \geqq \frac{1}{4\pi}\,.$$

Symmetry of C allows us to conclude that for $z \in B$, $z_1 \notin \langle 0, a\rangle$ and $|z_2| \leqq \sqrt{3}/2$ we can use (9) to estimate the integral in (8). Since the length of the corresponding part of B equals $2\pi/3$ we easily obtain (6).

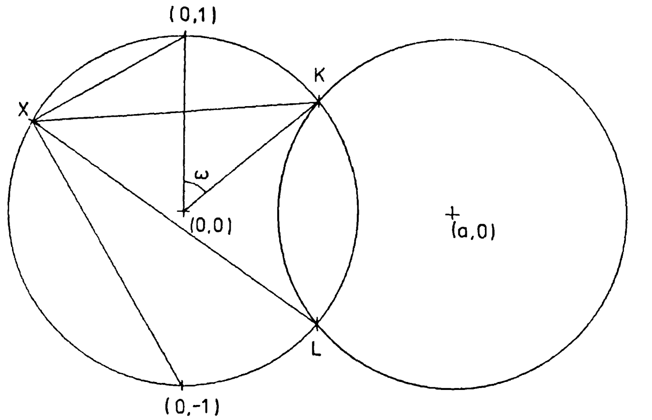
To prove (7) we can restrict ourselves to those $x \in B$ for which $|x| = 1$. Since $\tau_x = U\delta_x$, it may be easily verified that the function $x \to \|\tau_x\|$ is a lower semicontinuous function on B. Thus it suffices to prove (7) only for $x \in B$ such that $x_1 \in \in \langle -1, 0) \cup (0, a/2)$. [1], Proposition 2.8 yields

$$\|\tau_x\| = |2\, d_G(x) - 1| + 2\int_B |\varrho(x, z)|\, \mathrm{d}\varkappa_1(z) = 2\, v^G(x) = \\ = \frac{1}{\pi}\int_\Gamma n^G(\theta, x)\, \mathrm{d}\varkappa_1(\theta)\,.$$

For any $x \in B$, $x_1 < 0$ the ray $P(x, \theta)$ intersects B if and only if $x\,.\,\theta < 0$. Let us investigate $n^G(\theta, x)$ in this case. Clearly $1 \leqq n^G(\theta, x) \leqq 3$, and convexity of circles gives $n^G(\theta, x) = 1$ for the corresponding θ's and $y = (y_1, y_2) \in \partial\Omega_1(0)$ such that $y_1 < 0$ or $y_1 > a/2$ (see again the figure). Thus

$$\|\tau_x\| \leqq \frac{1}{\pi}\int_{M_1} 1\, \mathrm{d}\varkappa_1(\theta) + \frac{1}{\pi}\int_{M_2} 2\, \mathrm{d}\varkappa_1(\theta)\,,$$

where $M_1 = \{\theta \in \Gamma;\ x\,.\,\theta < 0\}$, $M_2 = \{\theta \in \Gamma;\ P(x, \theta) \cap \{y \in \partial\Omega_1(0);\ 0 \leqq y_1 \leqq \leqq a/2\} \neq \emptyset\}$.



For the first integral, the $\varkappa_1$-measure of the set M_1 is obviously equal to π and an elementary geometrical reasoning gives in the second case the value 2ω, where ω is the magnitude of the angle $\sphericalangle KXE$ (for the chords we have $KE = LF$), where $E = (0, 1)$, $F = (0, -1)$. Hence $\|\tau_x\| \leqq 1 + 1/8$. The same estimate can be established for $x \in B$ with $x_1 > a$.

Using the symmetry of B it is enough to proceed by analogy for $x \in B$ with $0 < < x_1 < a/2$ and $x_2 > 0$. It is easily seen that for such $x \in B$ for which $P(x, \theta)$ contains $y \in \partial\Omega_1(0)$, $y \neq x$, we have $n^G(\theta, x) = 1$ unless $y = (y_1, y_2)$ is such that $x_1 < y_1 < a/2$, $y_2 > 0$. For $\theta \in \Gamma$, $\theta\,.\,x > 0$ we have $n^G(\theta, x) \leqq 2$. Recall that the vector $(\sqrt{(1 - a^2/4)}, a/2)$ is a tangent vector of $\partial\Omega_1(a)$ at the point $(a/2, \sqrt{(1 - a^2/4)})$.

It is easily seen that $n^G(\theta, x) = 0$ unless $\theta_1 > 0$, $\theta_2 \leqq a/2$. Since $n^G(\theta, x) \leqq 3$ for all $\theta \in \Gamma$ we have

$$\|\tau_x\| \leqq \frac{1}{\pi}\int_{M_1} 1 \mathrm{d}\varkappa_1(\theta) + \frac{1}{\pi}\int_{M_2} 2 \,\mathrm{d}\varkappa_1(\theta) + \frac{1}{\pi}\int_{M_3} 2 \,\mathrm{d}\varkappa_1(\theta),$$

where $M_1 = \{\theta \in \Gamma;\ x \,.\, \theta < 0\}$, $M_2 = \{\theta \in \Gamma;\ P(x, \theta) \cap \{y \in \partial\Omega_1(0);\ y_2 > 0,\ x_1 < y_1 < a/2\} \neq \emptyset$, $M_3 = \{\theta \in \Gamma;\ \theta \,.\, x > 0,\ \theta_1 > 0,\ \theta_2 \leqq a/2\}$. Recall that $(\sqrt{(1 - a^2/4)}, -a/2)$ is a tangent vector of $\partial\Omega_1(0)$ at the point $(a/2, \sqrt{(1 - a^2/4)})$. We have

$$\|\tau_x\| \leqq 1 + \frac{2}{\pi}\varkappa_1(\{\theta \in \Gamma;\ \theta_1 > 0,\ \theta_2 \in (-a/2, a/2)\}) = 1 + \frac{8\omega}{\pi} \leqq 1 + \tfrac{1}{4}\,.$$

I would like to call reader's attention to the reprint by W. Winzell [3] that reached me after the completion of the present paper. B. Winzell characterized the domains with a smooth boundary (of class C^2), for which the operator U is contractive on $\mathscr{C}_0'(B)$.

References

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Souhrn

POZNÁMKA O KONTRAKTIVITĚ NEUMANNOVA OPERÁTORU

DAGMAR MEDKOVÁ

V článku se zkoumá Neumannův operátor na prostoru znaménkových měr na hranici, pro něž je míra prostoru rovna nule. Ukazuje se příklad množiny, jejíž doplněk je nekonvexní omezená množina a přitom příslušný Neumannův operátor je na tomto prostoru kontraktivní.

Резюме

ЗАМЕТКА О КОНТРАКТИВНОСТИ ОПЕРАТОРА НЕЙМАНА

DAGMAR MEDKOVÁ

В статье изучается оператор Неймана на пространстве зарядов на границе, для которых мера пространства равняется нулю. Приводится пример множества, дополнение которого является ограниченным невыпуклым множеством и притом соответствующий оператор Неймана является сжимающим на этом пространстве.

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