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EDGE-DISTANCE BETWEEN ISOMORPHISM CLASSES OF GRAPHS

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Summary. V. Kvasnička, V. Baláž, M. Sekanina and J. Koča have defined a distance (called here edge-distance) between isomorphism classes of graphs, based on the maximum number of edges of common subgraph. This paper concerns the graph whose vertex set is the set of all isomorphism classes of graphs with n vertices and in which two vertices are adjacent if and only if their edge-distance is equal to 1.

Keywords: Isomorphism classes of graphs, edge-distance of isomorphism classes of graphs.

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In [7] a distance between isomorphism classes of graphs was introduced. If $\mathfrak{G}_1$, $\mathfrak{G}_2$ are two isomorphism classes of graphs with n vertices, where n is a positive integer, and k is the maximum number of vertices of a graph which is isomorphic simultaneously to an induced subgraph of a graph from $\mathfrak{G}_1$ and to an induced subgraph of a graph from $\mathfrak{G}_2$, then the distance between $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is defined to be $n - k$. In the paper [8] a similar distance for isomorphism classes of trees was defined. Some similar metrics were studied by G. Chartrand, F. Saba and H.-B. Zou [1], F. Kaden [2] and F. Sobik [6]. Here we shall turn our attention to the metric defined by V. Kvasnička, V. Baláž, M. Sekanina and J. Koča [3]. In a certain sense this is an edge analogue of the metric introduced in [7].

Let $\mathfrak{G}_1$, $\mathfrak{G}_2$ be two isomorphism classes of graphs, let $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$. Let V_1 (or V_2) be the vertex set and E_1 (or E_2) the edge set of G_1 (or G_2, respectively). Let G_{12} be a graph which is isomorphic simultaneously to a subgraph of G_1 and to a subgraph of G_2 and has the maximum number of edges among all graphs with this property. Then the distance between $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is defined to be equal to $|E_1| + |E_2| - 2|E_{12}| + | |V_1| - |V_2| |$, where E_{12} is the edge set of G_{12}. As the authors of [3] assert, this distance has applications in chemistry. We shall call it the edge-distance.

Analogously as in [7], we take set $\mathscr{G}_n$ of all isomorphism classes of graphs with n vertices ($n \geqq 2$) and define the graph $G(\mathscr{G}_n)$ whose vertex set is $\mathscr{G}_n$ and in which two vertices are adjacent if and only if their edge-distance is equal to 1. We shall study this graph.

Theorem 1. *Let $\mathfrak{G}_1$, $\mathfrak{G}_2$ be two isomorphism classes from $\mathscr{G}_n$, let their edge-distance be* 1. *Then a graph $G_2 \in \mathfrak{G}_2$ is obtained from a suitable graph $G_1 \in \mathfrak{G}_1$ by adding or deleting one edge.*

Proof. We have $|E_1| + |E_2| - 2|E_{12}| = 1$. As $|E_{12}| \leqq |E_1|$, $|E_{12}| \leqq |E_2|$, this is possible if and only if either $|E_{12}| = |E_1| = |E_2| - 1$, or $|E_{12}| = |E_2| = |E_1| - 1$. Thus the graph G_{12} is isomorphic to one of the graphs G_1, G_2 and the other of these graphs is obtained from it by adding one edge. This implies the assertion.

This enables us to direct the edges of $G(\mathscr{G}_n)$ in such a way that an edge goes from $\mathfrak{G}_1$ to $\mathfrak{G}_2$ if and only if a graph from $\mathfrak{G}_2$ is obtained from a graph from $\mathfrak{G}_1$ by adding one edge. This graph $G \uparrow (\mathscr{G}_n)$ thus obtained is evidently acyclic. Its vertex set can be partitioned into sets of classes of graphs $\mathscr{H}_i$ for $i = 1, \ldots, \frac{1}{2}n(n-1)$ where $\mathscr{H}_i$ is the set of isomorphism classes of graphs from G_n having i edges. Each edge of $G \uparrow (\mathscr{G}_n)$ goes from a vertex of $\mathscr{H}_i$ into a vertex of $\mathscr{H}_{i+1}$ for some i.

For any two classes $\mathfrak{G}_1$, $\mathfrak{G}_2$ we may consider the graph G_{12} as having n vertices and thus belonging to G_n. If it has less vertices, we add the necessary number of isolated vertices to it. The symbol $\mathfrak{G}_{12}$ denotes the isomorphism class containing G_{12}.

Theorem 2. *The distance of any two vertices of $G(\mathscr{G}_n)$ (in the usual graph-theoretical sense) is equal to their edge-distance.*

Proof. Let $\mathfrak{G}_1$, $\mathfrak{G}_2$ be two vertices of $G(\mathscr{G}_n)$, let their edge-distance be k. Let $k_1 = |E_1| - |E_{12}|$, $k_2 = |E_2| - |E_{12}|$; evidently $k_1 + k_2 = k$, $|k_1 - k_2| = = | |E_1 - |E_2| |$. If $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$, then G_{12} (strictly speaking, a graph isomorphic to G_{12}) can be obtained by deleting k_1 edges from G_1 and G_2 can be obtained from G_{12} by adding k_2 edges. Thus in $G \uparrow (\mathscr{G}_n)$ there exists a directed path of the length k_1 from $\mathfrak{G}_{12}$ to $\mathfrak{G}_1$ and a directed path of the length k_2 from $\mathfrak{G}_{12}$ to $\mathfrak{G}_2$. These two paths have no common vertex except $\mathfrak{G}_{12}$; otherwise there would exist a graph with more edges than $|E_{12}|$ which would be isomorphic to a subgraph of G_1 and to a subgraph of G_2. Thus the union of these paths is (without considering the orientation of edges) a path of the length k connecting $\mathfrak{G}_1$ and $\mathfrak{G}_2$ in $G(\mathscr{G}_n)$ and the distance of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ in $G(\mathscr{G}_n)$ is less than or equal to their edge-distance. On the other hand, if there existed a path of a length $l < k$ connecting $\mathfrak{G}_1$ and $\mathfrak{G}_2$ in $G(\mathscr{G}_n)$, the graph G_2 could be obtained from G_1 by l operations of adding or deleting an edge. Let l_1 (or l_2) be the number of operations of deleting (or adding, respectively). Then $l = l_1 + l_2$, $l_1 - l_2 = |E_1| - |E_2| = k_1 - k_2$ and thus $k_1 - l_1 = k_2 - l_2$. If $l_1 \geqq k_1$ then also $l_2 \geqq k_2$ and vice versa; then $l = l_1 + l_2 \geqq k_1 + k_2 = k$, which is a contradiction. Thus $l_1 < k_1$, $l_2 < k_2$. As the ordering of these operations is evidently not substantial, we may first perform the operations of deleting and obtain a graph with $|E_1| - l_1 > |E_{12}|$ edges which would be isomorphic to a subgraph of G_1 and to a subgraph of G_2; this would be a contradiction. Thus the assertion is true.

Theorem 3. *The diameter of $G(\mathscr{G}_n)$ is $\frac{1}{2}n(n-1)$ and the unique pair of vertices* *of $G(\mathscr{G}_n)$ having the edge-distance $\frac{1}{2}n(n-1)$ consists of the class containing the complete graph and the class containing the graph consisting of isolated vertices.*

Proof. It is evident that the distance between the complete graph and the graph consisting of isolated vertices is equal to $\frac{1}{2}n(n-1)$. (Here, for the sake of brevity, we speak about the distance of graphs instead of the distance of isomorphism classes of graphs.) Now take two vertices $\mathfrak{G}_1$, $\mathfrak{G}_2$ of $G(\mathscr{G}_n)$. Let $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$, let the graphs G_1, G_2 have the same vertex set. Then the union of G_1 and G_2 cannot have more than $\frac{1}{2}n(n-1)$ edges. Their intersection has at most $|E_{12}|$ edges. As the number of edges of the union of G_1 and G_2 is equal to the sum of the numbers of edges of G_1 and G_2 minus the number of edges of their intersection, the distance between $\mathfrak{G}_1$ and $\mathfrak{G}_2$ cannot be greater than $\frac{1}{2}n(n-1)$. Evidently it can be equal to it only if one of the graphs G_1, G_2 has $\frac{1}{2}n(n-1)$ edges and the other has no edge.

Now we shall determine the radius of $G(\mathscr{G}_n)$. We use the concepts of the self-complementary graph and of the almost self-complementary graph.

A graph is called self-complementary, if it is isomorphic to its complement. These graphs were studied by G. Ringel [4] and H. Sachs [5]. A self-complementary, graph with n vertices exists if and only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. For the remaining cases we shall define an almost self-complementary graph.

An almost self-complementary graph is a graph which is isomorphic to a graph obtained from its complement by adding or omitting one edge.

We shall prove a theorem concerning these graphs.

Theorem 4. *An almost self-complementary graph with n vertices exists if and only if $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.*

Proof. Let such a graph G exist. From the definition it is clear that the union of G and its complement $\bar{G}$ has an odd number of edges. This union is the complete graph K_n with $\frac{1}{2}n(n-1)$ edges. The number $\frac{1}{2}n(n-1)$ is odd if and only if $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.

Now suppose that $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$ and construct an almost self-complementary graph with n vertices. If $n \equiv 2 \pmod 4$, we construct a self-complementary graph G' with $n-1$ vertices. We have $n - 1 \equiv 1 \pmod 4$ and thus, according to [4] and [5], there exists a vertex x of G' which is fixed in each isomorphic mapping of G' onto its complement $\bar{G}'$. Add a new vertex y to G' and join it by edges to exactly those vertices which are adjacent to x; the graph thus obtained will be denoted by G. Evidently G is isomorphic to the graph obtained from its complement $\bar{G}$ by deleting the edge xy. If $n \equiv 3 \pmod 4$, then let G' be a self-complementary graph with $n-2$ vertices and let x be again a vertex of G' fixed in each isomorphic mapping of G' onto its complement. Now we add two new vertices y and z to G' in the same way as the vertex y was added in the previous case. Moreover, we join y and z by an edge. The graph G thus obtained is again isomorphic to the graph obtained from its complement $\bar{G}$ by deleting the edge xy.

Theorem 5. *If* $n \equiv 0 \pmod 4$ *or* $n \equiv 1 \pmod 4$, *then the radius of* $G(\mathcal{G}_n)$ *is equal to* $\frac{1}{4}n(n-1)$ *and any class from* $\mathcal{G}_n$ *containing a self-complementary graph is its central vertex. If* $n \equiv 2 \pmod 4$ *or* $n \equiv 3 \pmod 4$, *then the radius of* $G(\mathcal{G}_n)$ *is equal to* $\frac{1}{4}n(n-1) + \frac{1}{2}$ *and any class from* $\mathcal{G}_n$ *containing an almost self-complementary graph is its central vertex.*

Proof. Let $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ and let G_1 be a self-complementary graph with n vertices; let $\mathfrak{G}_1$ be the isomorphism class containing it. Let $\mathfrak{G}_2$ be an arbitrary class from $\mathcal{G}_n$, let $G_2 \in \mathfrak{G}_2$ and suppose that G_2 has the same vertex set as G_1. Thus both G_1 and G_2 are subgraphs of the same complete graph K_n with n vertices. Each edge of K_n belongs either to G_1, or to its complement $\bar{G}_1$; hence this is true also for each edge of G_2. Let m be the number of edges of G_2. If at least $\frac{1}{2}m$ edges of G_2 belong to G_1, then the graph G_{12} (see the definition above) has at least $\frac{1}{2}m$ edges and the distance between G_1 and G_2 is at most $\frac{1}{4}n(n-1)$. If less than $\frac{1}{2}m$ edges of G_2 belong to G_1, then more than $\frac{1}{2}m$ edges of G_2 belong to $\bar{G}_1$ and, as $\bar{G}_1 \cong G_1$, we may proceed in the same way with $\bar{G}_1$ as with G_1. As the diameter of $G(\mathcal{G}_n)$ is $\frac{1}{2}n(n-1)$ and $\mathfrak{G}_1$ has the distance at most $\frac{1}{4}n(n-1)$ from any other vertex of $G(\mathcal{G}_n)$, the number $\frac{1}{4}n(n-1)$ is the radius of $G(\mathcal{G}_n)$. In the case when $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$ the proof is analogous.

Remark. Self-complementary graphs need not be unique central vertices of $G(\mathcal{G}_n)$. For example, for $n = 4$ the graph consisting of a triangle and an isolated vertex is a central vertex of $G(\mathcal{G}_4)$.

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Souhrn

HRANOVÁ VZDÁLENOST MEZI TŘÍDAMI ISOMORFISMU GRAFŮ

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V. Kvasnička, V. Baláž, M. Sekanina a J. Koča definovali jistou vzdálenost (která se zde nazývá hranová vzdálenost) mezi třídami isomorfismu grafů, která je založena na maximálním počtu hran společného podgrafu. Tento článek se zabývá grafem, jehož množinou uzlů je množina všech tříd isomorfismu grafů o n uzlech a v němž jsou dva uzly spojeny hranou právě tehdy, je-li jejich hranová vzdálenost rovna 1.

Резюме

РЕБЕРНОЕ РАССТОЯНИЕ МЕЖДУ КЛАССАМИ ИЗОМОРФИЗМА ГРАФОВ

BOHDAN ZELINKA

В. Квасничка, В. Балаж, М. Секанина и И. Коча определили расстояние (называемое здесь реберным расстоянием) между классами изоморфизма графов, основанное на максимальном числе ребер общего подграфа. В настоящей статье рассматривается граф, множеством вершин которого является множество всех классов изоморфизма графов с n вершинами и в котором две вершины смежны тогда и только тогда, когда их реберное расстояние равно 1.

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