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BOUNDS ON THE SERRE COHOMOLOGY OF PROJECTIVE VARIETIES

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Summary. An elementary method for giving bounds on the Serre cohomology of projective varieties is presented and some applications are given.

Keywords: Serre cohomology, bounds on Serre cohomology, cohomology groups of vector bundles, very ample divisors.

0. INTRODUCTION

This note presents a few results out of a list which will be published elsewhere in a more complete form. We want to show the usefulness of a very elementary method for giving bounds on the Serre cohomology of projective varieties. The same method applies to local cohomology of graded modules. The method bases on an idea we used in [1] to prove the finiteness of certain local cohomology modules. In [2] we showed how this method furnishes a lifting principle for the finiteness of (local or Serre-) cohomology. In this note we present a further development of the mentioned techniques. They furnish bounds on the cohomology groups of vector bundles similar to those of Elencwaig-Forster [3]. They also will give an estimate on the left vanishing order of the first cohomology of a very ample divisor on a normal complete variety X of dimension > 1. In fact, a more general statement will be given. As for terminology and notations see Hartshorne [5]. Least integer parts are denoted by $[\ ]$.

A. A GENERAL ESTIMATE

Let $X \subseteq \mathbb{P}^d_k$ be a projective scheme over an algebraically closed field k. We write $X = \operatorname{Proj}(A)$, where $A = \bigoplus_{n \geq 0} A_n$ is a graded homomorphic image of the polynomial ring $k[x_0, \ldots, x_d]$. Let $\mathscr{F}$ be a coherent sheaf over X, which we write as induced by a finitely generated graded A-module $M = \oplus M_n$: $\mathscr{F} = \tilde{M}$. Our goal is to give some bounds on the growth of the functions

$$h^i(X, \mathscr{F}(n)) = \dim_k (H^i(X, \mathscr{F}(n))\,.$$

Here $H^i(X, \mathscr{G})$ denotes the i-th Serre Cohomology of X with coefficients in a sheaf $\mathscr{G}$.

Let $f \in A_1$ be a form of degree 1. We write H_f for the corresponding hyperplane section $\operatorname{Proj}(A/fA)$ of X, and let $\iota\colon H_f \hookrightarrow X$ be the inclusion map. We say that f (or H_f) is *general with respect to* $\mathscr{F}$ if $H_f \cap \operatorname{Ass}(\mathscr{F}) = \emptyset$. We may assume that M has no torsion with respect to the maximal ideal $\mathfrak{m} = A_{>0}$ of A. Then f is general with respect to $\mathscr{F}$ iff f is regular with respect to M. Then, for all $n \in \mathbb{Z}$ we have short exact sequences

$$0 \to M(n) \xrightarrow{f} M(n+1) \to M(n+1)/f\,M(n) \to 0 .$$

Observing that $(M(n+1)/f\,M(n))^\sim = ((A/fA) \otimes_A M(n+1))^\sim = \iota_*(\mathscr{F}(n+1)|_{H_f}) = \iota_*(\mathscr{F}|_{H_f}(n+1))$ we thus get exact sequences

$$0 \to \mathscr{F}(n) \xrightarrow{f(n)} \mathscr{F}(n+1) \to \iota_*(\mathscr{F}|_{H_f}(n+1)) \to 0 .$$

Applying cohomology we get exact sequences:

$$(*)\quad \ldots H^{i-1}(H_f, \mathscr{F}|_{H_f}(n+1)) \to H^i(X, \mathscr{F}(n)) \xrightarrow{f = H^i(X, f(n))} H^i(H_f, \mathscr{F}|_{H_f}(n+1)) .$$

We now consider a linear system $\mathscr{H}$ of hyperplane sections, whose dimension is N. So we may write

$$\mathscr{H} = \{H_f \mid f \in V - \{0\}\} ,$$

where $V \subseteq A_1$ is a vector space of dimension $N + 1$. We say that $\mathscr{H}$ is in *general position* with respect to $\mathscr{F}$ if all $H_f \in \mathscr{H}$ are general with respect to $\mathscr{F}$.

We now define the numbers

$$r^i(n) = \max\{\dim_k(\ker[f\colon H^i(X, \mathscr{F}(n)) \to H^i(X, \mathscr{F}(n+1))]) \mid f \in V - \{0\}\} ,$$

$$s^i(n) = \max\{\dim_k(\operatorname{coker}[f\colon H^i(X, \mathscr{F}(n)) \to H^i(X, \mathscr{F}(n+1))]) \mid f \in V - \{0\}\} .$$

Clearly $r^i(n) \leqq h^i(X, \mathscr{F}(n))$, $s^i(n) \leqq h^i(X, \mathscr{F}(n+1))$.

(1) **Proposition.**

(i) $r^i(n) < h^i(X, \mathscr{F}(n)) \Rightarrow h^i(X, \mathscr{F}(n)) - h^i(X, \mathscr{F}(n+1)) \leqq r^i(n) - \left[\dfrac{N}{r^i(n)+1}\right]$.

(ii) $s^i(n) < h^i(X, \mathscr{F}(n+1)) \Rightarrow h^i(X, \mathscr{F}(n)) - h^i(X, \mathscr{F}(n+1)) \geqq$
$\geqq \left[\dfrac{N}{s^i(n)+1}\right] - s^i(n)$.

Proof. (Sketch) only (i): Write $U = H^i(X, \mathscr{F}(n))$, $W = H^i(X, \mathscr{F}(n+1))$. We find $f_0, \ldots, f_N \in A_1$ such that $f_0, \ldots, f_N$ is a basis of V. Fix a k-base of U and a k-base of W and let $M_i = (m^{(i)}_{\nu\mu})$ be the matrix which corresponds to the linear map $f_i\colon U \to W$. Put $u = h(X, \mathscr{F}(n)) = \dim U$, $w = \dim h(X, \mathscr{F}(n+1))$. M_i is a $u \times w$-matrix. Let $f \in V - \{0\}$. Then we write $f = \sum \alpha_i f_i$. $M(\alpha) = \sum \alpha_i M_i$ is the matrix which corresponds to $f\colon U \to W$. $M(\alpha)$ is a matrix of rank $\geqq u - r^i(n)$. Let $T_0, \ldots, T_N$

be indeterminates and let $\boldsymbol{T} \subseteq k[T_0, \ldots, T_N] =: B$ be the ideal spanned by the $(u - r^i(n)) \times (u - r^i(n))$-minors of $M(\boldsymbol{T}) = \sum T_i M_i$. $\boldsymbol{T}$ is a homogeneous ideal and

$$\operatorname{codim}_{\mathbb{P}^N}(V_+(I)) \leqq (w - u + r^i(n) + 1)(r^i(n) + 1),$$

where $V_+(I) \subseteq \mathbb{P}^N = \operatorname{Proj}(B)$ is the projective set defined by I. As $M(\alpha)$ is of rank $\geqq u - r^i(n)$, we have $V_+(I) = \emptyset$. From this we get

$$N < (w - u + r^i(n) + 1)(r^i(n) + 1),$$

thus

$$u - w \leqq r^i(n) - \left[\frac{N}{r^i(n) + 1}\right].$$

To apply this result we first introduce

$$h^i(\mathcal{H}, \mathcal{F}|_{\mathcal{H}}(n)) =_{\mathrm{Def.}} \min \{h^i(H_f, \mathcal{F}|_{H_f}(n) \mid f \in V - \{0\}\}.$$

In fact this value is *generic*, e.g. attained for an open set of members H_f of $\mathcal{H}$. We introduce the *left* (right, respectively) *vanishing order* for H^i *on* $\mathcal{F}$ as:

$$n^i_{\mathcal{F}} = \inf \{n \in \mathbb{Z} \mid H^i(X, \mathcal{F}(n + 1)) \neq 0\},$$

$$m^i_{\mathcal{F}} = \sup \{m \in \mathbb{Z} \mid H^i(X, \mathcal{F}(n - 1)) \neq 0\} \, (\in \mathbb{Z} \cup \{\pm\infty\}),$$

thereby using the convention $\inf(\emptyset) = \infty$, $\sup\,(\emptyset) = -\infty$. Moreover, we put $u^+ = \max\{0, u\}$ $(u \in \mathbb{R})$ and set

$$n^i_{\mathcal{F}|\mathcal{H}} = \inf \{n^i_{\mathcal{F}|H_f} \mid H_f \in \mathcal{H}\}.$$

$$m^i_{\mathcal{F}|\mathcal{H}} = \sup \{m^i_{\mathcal{F}|H_f} \mid H_f \in \mathcal{H}.$$

Then, using the exact sequences (*) and (1) we get

(2) **Proposition.** *Let $\mathcal{H}$ be general with respect to $\mathcal{F}$ and let $i \geqq 0$, $n_0 \in \mathbb{Z}$. Then*

$$h^i(X, \mathcal{F}(n)) \leqq \begin{cases} [h^i(X, \mathcal{F}(n_0)) + \sum\limits_{n < m \leqq n_0} h^{i-1}(\mathcal{H}, \mathcal{F}|_{\mathcal{H}}(m)) - \\ - (\min\{n_0, n^{i-1}_{\mathcal{F}|\mathcal{H}}\} - n)^+ N]^+, \\ \textit{if} \quad i > 0 \quad \textit{and} \quad n < n_0; \\ [h^i(X, \mathcal{F}(n_0)) + \sum\limits_{n_0 < m \leqq n} h^i(\mathcal{H}, \mathcal{F}|_{\mathcal{H}}(m)) - \\ - (n - \max\{n_0, m^i_{\mathcal{F}|\mathcal{H}} - 1\})^+ N]^+, \\ \textit{if} \quad i \geqq 0, \quad n \geqq n_0. \end{cases}$$

(3) Remark. The importance of this result is that it gives bounds on the function $h^i(X, \mathcal{F}(\cdot))$ in terms of its value for a particular argument n_0 and in terms of the behaviour of the cohomology of restrictions to the hyperplanes in $\mathcal{H}$. In an earlier paper we gave a much less specific bound in the case $N = 1$ which already turned out to be useful. Namely, it provided an immediate proof of the vanishing theorem of Severi-Zariski Serre, (cf. [2]).

To test our estimate we give an application to vector bundles over $\mathbb{P}_k^d$. We start with some notations: $\mathbb{Z}^+$ denotes the set of non-negative integers. We put:

$$\begin{aligned}
B &= \{s\colon \mathbb{Z} \to \mathbb{Z}^+\}\,,\\
B^+ &= \{s \in B \mid s(n) = 0,\ \forall n \gg 0\}\,,\\
B^- &= \{s \in B \mid s(n) = 0,\ \forall n \ll 0\}\,,\\
B^0 &= B^+ \cap B^-\,.
\end{aligned}$$

Moreover, if $s \in B$, we define

$$\nu_s = \inf\{n \in \mathbb{Z} \mid s(n+1) \neq 0\}\,,\quad \mu_s = \sup\{n \in \mathbb{Z} \mid s(n-1) \neq 0\}\,.$$

Then, for $\varrho \in \mathbb{Z}^+$ we define two operators

$$\begin{aligned}
&T_\varrho\colon B^+ \to B^+\,,\quad U_\varrho\colon B^- \to B^- \quad \text{by}\\
&T_\varrho s(n) = \Big[\sum_{n<m} s(m) - (\nu_s - n)^+ \varrho\Big]^+\,,\quad (s \in B^+)\,,\\
&U_\varrho s(n) = \Big[\sum_{m \leqq n} s(m) - (n - \mu_s + 1)^+ \varrho\Big]^+\,,\quad (s \in B^-)\,,
\end{aligned}$$

Note that

$$\begin{aligned}
&T_\varrho, U_\varrho\colon B^0 \to B^0 \quad \text{if} \quad \varrho > 0\,,\\
&T_\varrho(0) = U_\varrho(0) = 0\,.
\end{aligned}$$

Now, let $d > 1$ and let $\mathscr{E}$ be a bundle over $\mathbb{P}_k^d$ which is of generic splitting type $(a_1, \ldots, a_r) = (a) \in \mathbb{Z}^r (a_1 \geqq a_2 \ldots \geqq a_r)$. Let c_1, c_2 be the first two Chern classes of $\mathscr{E}$.

Finally, introduce the *generic span* of $\mathscr{E}$, defined as

$$\sigma = a_1 - a_r\,.$$

(3) **Lemma.** *There is a bound $\delta = \delta(\sigma, c_1, c_2)$ depending only on σ, c_1 and c_2 such that each linearly embedded projective plane $\mathbb{P}^2 \hookrightarrow \mathbb{P}^d$ satisfies*

$$h^1(\mathbb{P}^2, \mathscr{E}|_{\mathbb{P}^2}) \leqq \delta\,.$$

Proof. As σ, c_1 and c_2 are not affected under restriction to projective planes, we may put $d = 2$. But then we may complete the proof by the Riemann-Roch theorem for bundles, cf. [3].

Then, introduce the functions

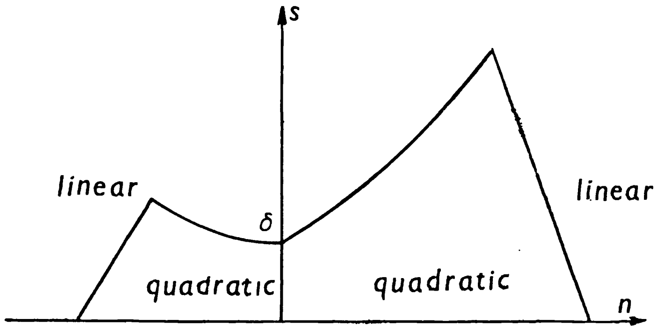
$$\begin{aligned}
p(n) &= (r(n + 1 + \sigma) - c_1)^+ \in B^-\,,\\
g(n) &= (-r(n + 1 + \sigma) + c_1)^+ \in B^+\,,
\end{aligned}$$

and

$$s(n)=\begin{cases}\Big[\delta+\sum\limits_{n\leqq m\leqq 0}p(m)-(-p(1)-\delta-1-n)^{+}\,2\Big]^{+} & \text{if}\quad n\leqq 0\,,\\ \Big[\delta+\sum\limits_{0<m\leqq n}q(m)-(q(0)+\delta+n)^{+}\,2\Big]^{+} & \text{if}\quad n>0\,.\end{cases}$$

Clearly s belongs to B^0 and has a graph of the type sketched below.

Using these notations we get:



(4) **Proposition.**

(i) $$\begin{cases} h^0(\mathbb{P}^d,\mathscr{E}(n))\leqq \sum\limits_{j:a_j+n\geqq 0}\binom{n+a_j+d}{d} & (\leqq t(\sigma,c_1;n))\,,\\ h^d(\mathbb{P}^d,\mathscr{E}(n))\leqq \sum\limits_{j:a_j+n<-d}\binom{-n-a_j-1}{d} & (\leqq \bar{t}(\sigma,c_1;n))\,;\end{cases}$$

(ii) $$\underset{(0<i<d)}{h^i(\mathbb{P}^d,\mathscr{E}(n))}\leqq \underbrace{T_{d-2}\circ\ldots\circ T_{d-2}}_{i+1}\circ\underbrace{U_{d-i-1}\circ\ldots\circ U_1}_{d-i-1}\,s(n)\,.$$

(5) Remark. $s(n)$ is piecewise polynomial in n. Moreover, its coefficients are polynomials in σ, c_1, c_2. So our result extends a theorem of Forster-Elencwaig, [3], which shows that $h^i(\mathbb{P}^d,\mathscr{E})$ is bounded by a (polynomial) estimate depending only on c_1, c_2 and σ. If $\mathscr{E}$ is stable, σ may be estimated by c_1 and c_2. Then the bounds depend only on c_1, c_2.

C. VERY AMPLE DIVISORS

We choose $X\subseteq\mathbb{P}^d_k$ as reduced and of pure dimension $t>1$. Then the set of all closed points $x\in X$ for which the first local cohomology $H^1_{m_{X,x}}(\mathcal{O}_{X,x})\neq 0$, is finite:

$$Z:=\{x\in X\mid x \text{ closed},\ H^1_{m_{X,x}}(\mathcal{O}_{X,x})\neq 0\}=\{x_1,\ldots,x_r\}\,.$$

Moreover, we have

$$\mu = \mu(X) := \sum_{i=1}^{r} (H^1_{m_{X,x_i}}(\mathcal{O}_{X,x_i})) < \infty .$$

If X is normal (or – more generally – satisfies S_2), then $\mu(X) = 0$. Now it is easy to verify that

$$h^1(X, \mathcal{O}_X(n)) = h^1(\mathcal{O}_X(n)) = \mu(X) \quad \text{for all} \quad n \ll 0 .$$

Our goal is to find a bound on n for which this equality holds. To give such a bound we introduce the following number:
$\operatorname{depth}'(X) = \min\{i > 1 \mid H^i_{m_{X,x}}(\mathcal{O}_{X,x}) \neq 0 \quad \text{for some closed point} \quad x \in X\} \quad (>1)$.
Then we have

(6) **Proposition.** *Let X be reduced and of pure dimension >1. Assume that $X - Y$ is connected for each closed set $Y \subseteq X$ with* $\dim(Y) = 1$.
Then

(i) $\mu \leqq h^1(\mathcal{O}_X(n)) \leqq \max\{\mu, h^1(\mathcal{O}_X(n+1) - \operatorname{depth}'(X) + 1\}, \quad n < 0;$

(ii) $h^1(\mathcal{O}_X(n)) = \mu(X)$ *for all* $n \leqq -\dfrac{h^1(\mathcal{O}_X) - \mu}{\operatorname{depth}'(X) - 1}$.

(7) **Corollary.** *Let X be an irreducible complete variety of dimension >1 and let $\mathscr{L}$ be a very ample invertible sheaf on X. Then*

$$h^1(X, \mathscr{L}^n) = \mu \quad \textit{for all} \quad n \leqq \left[-\frac{h^1(\mathcal{O}_X) - \mu}{\operatorname{depth}'(X) - 1} \right].$$

We give an idea of the proof of (6) in case when X is an irreducible surface. In this case $\operatorname{depth}'(X) = 2$. By a reduction argument, which we do not give here, we may restrict ourselves to the case when $\mu(X) = 0$. We choose $V \subseteq A_1$ as a k-space dimension 2 such that $X \cap H_f$ is reduced and connected (this is possible by a Bertini argument, [4]) for all $f \in V - \{0\}$. Then (*) gives rise to sequences

$$0 \to \underset{=0}{H^0(\mathcal{O}_X(-1))} \to \underset{=k}{H^0(\mathcal{O}_X)} \xrightarrow{\simeq} \underset{=k}{H^0(\mathcal{O}_{H_f})} \to H^1(\mathcal{O}_X(-1)) \xrightarrow{f} H^1(\mathcal{O}_X)$$

and

$$\text{if} \quad n < 0: \quad \underset{\cong 0}{H^0(\mathcal{O}_{H_f}(n))} \to H^1_x(\mathcal{O}_X(n-1)) \xrightarrow{f} H^1(\mathcal{O}_X(n))$$

So, for all $n < 0$ we have $r^1(n) = 0$. As $N = 1$ we thus get by (1) for all $n < 0$ with $h^1(\mathcal{O}_X(n)) > 0$ the inequality $h^1(\mathcal{O}_X(n)) \leqq h^1(\mathcal{O}_X(n+1)) - \left[\frac{1}{1}\right] = h^1(\mathcal{O}_X(n+1)) - 1$.
If $h^1(\mathcal{O}_X(n)) = 0$ for some $n < 0$, then $r^1(n-1) = 0$ shows that $h^1(\mathcal{O}_X(n-1)) = 0$.
This proves our claim.

As an application of (7) we get

(8) **Corollary.** *Let X be an irreducible complete variety of dimension >1 which satisfies the second Serre condition S_2. Then, for each very ample invertible sheaf $\mathscr{L}$ on X we have*

$$H^1(X, \mathscr{L}^n) = 0\,, \quad \text{if} \quad n \leqq -h^1(X, \mathcal{O}_X)\,.$$

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Souhrn

MEZE PRO SERREOVU KOHOMOLOGII PROJEKTIVNÍCH VARIET

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Je podána elementární metoda získání mezí pro Serreovu kohomologii projektivních variet a jsou ukázány některé její aplikace.

Резюме

ГРАНИЦЫ ДЛЯ КОГОМОЛОГИЙ СЕРРА ПРОЕКТИВНЫХ МНОГООБРАЗИЙ

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В статье изложен элементарный метод для получения границ для когомологий Серра проективных многообразий и указаны некоторые его приложения.

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