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PROJECTION AND COVERING IN A SET WITH ORTHOGONALITY

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Summary. We consider a set with orthogonality $(\Omega, \perp)$ and the corresponding complete lattice with orthogonality $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$. If the lattice $\mathscr{S}$ is orthomodular and if for every $x \in \Omega$, $x \neq o$, the set $\{x\}^{\perp\perp}$ is an atom in $\mathscr{S}$, then the following statements are equivalent.

(i) If $A \in S$, $x \in \Omega$, $x \notin A$, then $A \vee \{x\}^{\perp\perp}$ covers the set A.

(ii) If $x \in \Omega$, $A \in S$, $x \notin A$, $x \notin A^{\perp}$, then there are atoms $A_1 \subset A$ and $A_2 \subset A^{\perp}$ such that $x \in A_1 \vee A_2$.

(iii) If $A_1, A_2 \in S$, $A_1 \perp A_2$, $A_1 \neq \{o\} \neq A_2$, $x \in A_1 \vee A_2$, $x \notin A_1$, $x \notin A_2$, then there are elements $x_1 \in A_1$ and $x_2 \in A_2$ such that $x \in \{x_1\}^{\perp\perp} \vee \{x_2\}^{\perp\perp}$.

(iv) If $A \in S$, $x \in \Omega$, $x \notin A$, $x \notin A^{\perp}$, then $A \cap (A^{\perp} \vee \{x\}^{\perp\perp})$ and $A^{\perp} \cap (A \vee \{x\}^{\perp\perp})$ are atoms in $\mathscr{S}$.

Keywords: set with orthogonality, orthomodular lattice, set B covers set A, atoms in a set with orthogonality.

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1. The present paper develops some ideas of [1] and brings three statements which are equivalent to the "projection" axiom of [2] (see Axiom III of [2] on p. 279 and Axiom P below). We consider a set with an orthogonality relation $(\Omega, \perp)$. Let us recall that a relation $\perp \subset \Omega \times \Omega$ is called an orthogonality relation if 1. $\perp$ is symmetric, 2. there is a distinguished element o such that $\{o\} \times \Omega \subset \perp$ and the intersection of $\perp$ with the diagonal is exactly (o, o). The presence of an orthogonality relation $\perp$ on the set Ω gives rise to a complete lattice $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$ where S is the family of all subsets A of Ω satisfying $A = (A^{\perp})^{\perp}$. The set Ω plays the role of the unit element and $\{o\}$ plays the role of the nought element.

2. In what follows, we shall apply the following equivalent conditions on a lattice with an orthogonality relation $\mathscr{P} = (P, \leqq, \perp, 1, 0)$:

2.1. $\mathscr{P}$ is orthomodular.

2.2. If $a, b \in P$, $a \leqq b$, then $b = a \vee (a^{\perp} \wedge b)$.

2.3. If $a, b \in P$, $a \leqq b$, $a^{\perp} \wedge b = 0$, then $a = b$.

2.4. If $a, b, c \in P$, $a \leqq c$, $b \leqq c^{\perp}$, then $(a \vee b) \wedge c = a$.

If $(\Omega, \perp)$ is a set with an orthogonality relation and $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$ is the corresponding complete lattice with orthogonality, we shall consider the following axioms.

2.5. Axiom A. *For every $x \in \Omega$, $x \neq o$, $\{x\}^{\perp\perp}$ is an atom in $\mathscr{S}$.*

Let $A, B \in S$, $A \neq B$. We say that the set B covers the set A and write $A \prec B$, if $C \in S$, $A \subset C \subset B$ imply either $A = C$ or $B = C$.

2.6. Axiom C. *If $A \in S$, $x \in \Omega$, $x \notin A$, then $A \prec A \vee \{x\}^{\perp\perp}$.*

2.7. Axiom P. *If $x \in \Omega$, $A \in S$, $x \notin A$, $x \notin A^{\perp}$, then there exist atoms $A_1 \subset A$ and $A_2 \subset A^{\perp}$ such that $x \in A_1 \vee A_2$.*

2.8. Axiom Q. *If $A_1, A_2 \in S$, $A_1 \perp A_2$, $A_1 \neq \{o\} \neq A_2$, $x \in A_1 \vee A_2$, $x \notin A_1$, $x \notin A_2$, then there exist elements $x_1 \in A_1$ and $x_2 \in A_2$ such that $x \in \{x_1\}^{\perp\perp} \vee \{x_2\}^{\perp\perp}$.*

2.9. Axiom R. *If $A \in S$, $x \in \Omega$, $x \notin A$, $x \notin A^{\perp}$, then $A \cap (A^{\perp} \vee \{x\}^{\perp\perp})$ and $A^{\perp} \cap \cap (A \vee \{x\}^{\perp\perp})$ are atoms in $\mathscr{S}$.*

The main result of this paper is obtained in the following theorem.

2.10. Theorem. *If the lattice $\mathscr{S}$ is orthomodular and satisfies Axiom A, then Statements* 2.6, 2.7, 2.8 *and* 2.9 *are equivalent.*

Proof. 1. 2.6 $\Rightarrow$ 2.7. Let $A \in S$, $x \in \Omega$, $x \notin A$, $x \notin A^{\perp}$. Then we have $A \prec A \vee \vee \{x\}^{\perp\perp}$, $A^{\perp} \prec A^{\perp} \vee \{x\}^{\perp\perp}$. According to Statement 2.3 it is true that $A^{\perp} \cap \cap (A \vee \{x\}^{\perp\perp}) \neq \{o\} \neq A \cap (A^{\perp} \vee \{x\}^{\perp\perp})$. Let us suppose that $A_1 \in S$ is an atom and let $\{o\} \neq A_1 \subset A \cap (A^{\perp} \vee \{x\}^{\perp\perp})$. Then $A_1 \subset A^{\perp} \vee \{x\}^{\perp\perp}$, hence $A^{\perp} \subset \subset A^{\perp} \vee A_1 \subset A^{\perp} \vee \{x\}^{\perp\perp}$. If $A^{\perp} = A^{\perp} \vee A_1$, then $A_1 \subset A^{\perp}$, which implies $A_1 \subset \subset A^{\perp} \cap A = \{o\}$ – a contradiction. Thus, $A^{\perp} \vee A_1 = A^{\perp} \vee \{x\}^{\perp\perp}$. In accordance with Statement 2.4, we have $A_1 = (A^{\perp} \vee A_1) \cap A = A \cap (A^{\perp} \vee \{x\}^{\perp\perp})$. It can be shown in a similar way that $A_2 = A^{\perp} \cap (A \vee \{x\}^{\perp\perp})$ is an atom. Moreover, $A_1 \subset A$, $A_2 \subset A^{\perp}$. Since $A_1 \vee A_2 \subset A \vee A_2$, we have, in accordance with Statement 2.2, that $A_1 \vee A_2 = (A \vee A_2) \cap [(A \vee A_2)^{\perp} \vee A_1 \vee A_2]$. Since $A_2 \subset A^{\perp}$, by the same statement, we have $A^{\perp} = A_2 \vee (A_2^{\perp} \cap A^{\perp}) = A_2 \vee (A_2 \vee A)^{\perp}$. Hence $A_1 \vee A_2 = (A \vee A_2) \cap (A^{\perp} \vee A_1)$. Again by Statement 2.2, we have $A_1 \vee A^{\perp} = = [A \cap (A^{\perp} \vee \{x\}^{\perp\perp})] \vee A^{\perp} = A^{\perp} \vee \{x\}^{\perp\perp}$ because $A^{\perp} \subset A^{\perp} \vee \{x\}^{\perp\perp}$. Analogously, $A_2 \vee A = A \vee \{x\}^{\perp\perp}$. Thus $A_1 \vee A_2 = (A_1 \vee A^{\perp}) \cap (A_2 \vee A) = = (A^{\perp} \vee \{x\}^{\perp\perp}) \cap (A \vee \{x\}^{\perp\perp}) \supset \{x\}^{\perp\perp}$ which yields $x \in A_1 \vee A_2$.

2. 2.7 $\Rightarrow$ 2.6. Let us suppose that $A \in S$, $x \in \Omega$, $x \notin A$. Since $A \subset A \vee \{x\}^{\perp\perp}$, according to Statement 2.2, we have $A \vee \{x\}^{\perp\perp} = A \vee [A^{\perp} \cap (A \vee \{x\}^{\perp\perp})]$. Let us denote $\tilde{A}_2 = A^{\perp} \cap (A \vee \{x\}^{\perp\perp})$. If $A^{\perp} \cap (A \vee \{x\}^{\perp\perp}) = \{o\}$, in accordance with Statement 2.3, we have $A = A \vee \{x\}^{\perp\perp}$, hence $x \in A$ – a contradiction. Then $\tilde{A}_2 \neq \{o\}$. Moreover, let $x \notin A^{\perp}$. Then our assumption implies that there are atoms $A_1 \subset A$ and $A_2 \subset A^{\perp}$ such that $\{x\}^{\perp\perp} \subset A_1 \vee A_2$. We have $(\{x\}^{\perp\perp} \vee A) \cap A^{\perp} \subset \subset (A_1 \vee A_2 \vee A) \cap A^{\perp} = A_2$, where the last identity follows from Statement 2.4. Then $\tilde{A}_2 = A_2$. Let $B \in S$, $A \subset B \subset A \vee \{x\}^{\perp\perp}$ and $A \neq B$. By Statement 2.3 we

have $\{o\} \neq A^\perp \cap B \subset A^\perp \cap (A \vee \{x\}^{\perp\perp}) = A_2$, hence $\{o\} \neq A^\perp \cap B = A^\perp \cap$ $\cap (A \vee \{x\}^{\perp\perp})$. According to Statement 2.2, $B = A \vee (A^\perp \cap B) = A \vee [A^\perp \cap$ $\cap (A \vee \{x\}^{\perp\perp})] = A \vee \{x\}^{\perp\perp}$. Now, if $x \in A^\perp$, then $(\{x\}^{\perp\perp} \vee A) \cap A^\perp = \{x\}^{\perp\perp}$. If again $B \in S$, $A \subset B \subset A \vee \{x\}^{\perp\perp}$ and $A \neq B$, by Statement 2.3, we have $\{o\} \neq$ $\neq A^\perp \cap B \subset A^\perp \cap (A \vee \{x\}^{\perp\perp}) = \{x\}^{\perp\perp}$, hence $\{o\} \neq A^\perp \cap B = A^\perp \cap (A \vee$ $\vee \{x\}^{\perp\perp})$. Then, in accordance with Statement 2.2, $B = A \vee (A^\perp \cap B) = A \vee$ $\vee [A^\perp \cap (A \vee \{x\}^{\perp\perp})] = A \vee \{x\}^{\perp\perp}$. Summarizing, the lattice $\mathcal{S}$ satisfies Axiom C.

3. 2.7 $\Rightarrow$ 2.9. Let $A \in S$, $x \in \Omega$, $x \notin A$, $x \notin A^\perp$. Then there exist atoms $A_1 \subset A$ and $A_2 \subset A^\perp$ such that $x \in A_1 \vee A_2$. According to Lemma 2.8 in [1], we have $A_1 = A \cap (A^\perp \vee \{x\}^{\perp\perp})$, $A_2 = A^\perp \cap (A \vee \{x\}^{\perp\perp})$.

4. 2.9 $\Rightarrow$ 2.7. Let again $A \in S$, $x \in \Omega$, $x \notin A$, $x \notin A^\perp$. Denote $A \cap (A^\perp \vee \{x\}^{\perp\perp}) =$ $= A_1$, $A^\perp \cap (A \vee \{x\}^{\perp\perp}) = A_2$. It was shown in part 1 of this proof that $x \in A_1 \vee A_2$.

5. 2.7 $\Rightarrow$ 2.8. Let $A_1, A_2 \in S$, $A_1 \neq \{o\} \neq A_2$, $A_1 \perp A_2$, $x \in A_1 \vee A_2$, $x \notin A_1$, $x \notin A_2$. Denote $A = A_1 \vee A_2$. If $x \in A_2^\perp$, then Statement 2.4 yields $x \in A \cap A_2^\perp =$ $= (A_1 \vee A_2) \cap A_2^\perp = A_1$ – a contradiction. Similarly, $x \notin A_1^\perp$. Since $x \notin A_1$, $x \notin A_1^\perp$, in accordance with Axiom P, there are $x_1 \in A_1$ and $x_2 \in A_1^\perp$ such that $x \in \{x_1\}^{\perp\perp} \vee \{x_2\}^{\perp\perp}$. In comparison with part 3 of this proof, we get $\{x_1\}^{\perp\perp} =$ $= (A_1^\perp \vee \{x\}^{\perp\perp}) \cap A_1$, $\{x_2\}^{\perp\perp} = (A_1 \vee \{x\}^{\perp\perp}) \cap A_1^\perp$. The fact $A_2 \subset A_1^\perp$ implies $(A_1 \vee \{x\}^{\perp\perp}) \cap A_2 \subset (A_1 \vee \{x\}^{\perp\perp}) \cap A_1^\perp$. If $\{o\} = (A_1 \vee \{x\}^{\perp\perp}) \cap A_2$, then $A_1 =$ $= [(A_1 \vee \{x\}^{\perp\perp}) \cap A_2] \vee A_1$. We have $[A_1^\perp \cap (A_1 \vee \{x\}^{\perp\perp}) \cap (A_1 \vee A_2)] \vee A_1 =$ $= [(A_1 \vee \{x\}^{\perp\perp}) \cap A_2] \vee A_1$. Since $A_1 \subset (A_1 \vee \{x\}^{\perp\perp}) \cap (A_1 \vee A_2)$, by virtue of Statement 2.2, we have $(A_1 \vee \{x\}^{\perp\perp}) \cap (A_1 \vee A_2) = A_1 \vee [A_1^\perp \cap (A_1 \vee$ $\vee \{x\}^{\perp\perp}) \cap (A_1 \vee A_2)]$, which yields $(A_1 \vee \{x\}^{\perp\perp}) \cap A = A_1 \vee [(A_1 \vee \{x\}^{\perp\perp}) \cap$ $\cap A_2]$ in accordance with Statement 2.4. Hence $A_1 = [(A_1 \vee \{x\}^{\perp\perp}) \cap A_2] \vee$ $\vee A_1 = (A_1 \vee \{x\}^{\perp\perp}) \cap A = A_1 \vee \{x\}^{\perp\perp}$. Therefore, $x \in A_1$ – a contradiction. Thus $\{o\} \neq (A_1 \vee \{x\}^{\perp\perp}) \cap A_2$ which implies $\{x_2\}^{\perp\perp} = (A_1 \vee \{x\}^{\perp\perp}) \cap A_2$.

6. 2.8 $\Rightarrow$ 2.7. It sufices to put $A_1 = A \in S$ and $A_2 = A^\perp$. The proof of our theorem is complete.

2.11. Remark. It was shown in part 5 of the proof of Theorem 2.10 that $\{x_1\}^{\perp\perp} =$ $= (A_1^\perp \vee \{x\}^{\perp\perp}) \cap A_1$ and that $\{x_2\}^{\perp\perp} = (A_1 \vee \{x\}^{\perp\perp}) \cap A_1^\perp = (A_1 \vee \{x\}^{\perp\perp}) \cap$ $\cap A_2$. Moreover, we shall show that $\{x_1\}^{\perp\perp} = (A_2 \vee \{x\}^{\perp\perp}) \cap A_1 = (A_2 \vee$ $\vee \{x\}^{\perp\perp}) \cap A_2^\perp$ and $\{x_2\}^{\perp\perp} = (A_2^\perp \vee \{x\}^{\perp\perp}) \cap A_2$.

Proof. 1. The inclusion $A_2 \subset A_1^\perp$ implies $(A_2 \vee \{x\}^{\perp\perp}) \cap A_1 \subset (A_1^\perp \vee \{x\}^{\perp\perp}) \cap$ $\cap A_1 = \{x_1\}^{\perp\perp}$. If $\{o\} = (A_2 \vee \{x\}^{\perp\perp}) \cap A_1$, hence $A_2 = [(A_2 \vee \{x\}^{\perp\perp}) \cap A_1] \vee$ $\vee A_2 = (A_2 \vee \{x\}^{\perp\perp}) \cap A = A_2 \vee \{x\}^{\perp\perp}$, which can be proved in a similar way as in part 5 of the proof of Theorem 2.10. Hence $x \in A_2$ – a contradiction. Thus $\{o\} \neq (A_2 \vee \{x\}^{\perp\perp}) \cap A_1 = (A_1^\perp \vee \{x\}^{\perp\perp}) \cap A_1$.

2. By virtue of the part 5 of the proof of Theorem 2.10 we have $x \notin A_2^\perp$. In comparison with the part 3 of the proof of Theorem 2.10, we have $x \in \{x_3\}^{\perp\perp} \vee \{x_4\}^{\perp\perp}$ where $\{x_3\}^{\perp\perp} = (A_2^\perp \vee \{x\}^{\perp\perp}) \cap A_2 \supset (A_1 \vee \{x\}^{\perp\perp}) \cap A_2 = \{x_2\}^{\perp\perp}$ and $\{x_4\}^{\perp\perp} =$ $= (A_2 \vee \{x\}^{\perp\perp}) \cap A_2^\perp \supset (A_2 \vee \{x\}^{\perp\perp}) \cap A_1 = \{x_1\}^{\perp\perp}$. Our assertion is proved.

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Souhrn

PROJEKCE A POKRYTÍ V MNOŽINĚ S ORTOGONALITOU

JAN HAVRDA

Uvažuje se množina s ortogonalitou $(\Omega, \perp)$ a jí odpovídající úplný svaz s ortogonalitou $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$. Je-li $\mathscr{S}$ ortomodulární a jestliže pro každé $x \in \Omega$, $x \neq o$, je $\{x\}^{\perp\perp}$ atom v $\mathscr{S}$, pak následující tvrzení jsou ekvivalentní.

(i) Je-li $A \in S$, $x \in \Omega$, $x \notin A$, pak $A \vee \{x\}^{\perp\perp}$ pokrývá A.

(ii) Je-li $x \in \Omega$, $A \in S$, $x \notin A$, $x \notin A^{\perp}$, pak existují atomy $A_1 \subset A$ a $A_2 \subset A^{\perp}$ tak, že $x \in A_1 \vee A_2$.

(iii) Jsou-li $A_1, A_2 \in S$, $A_1 \perp A_2$, $A_1 \neq \{o\} \neq A_2$, $x \in A_1 \vee A_2$, $x \notin A_1$, $x \notin A_2$, pak existují prvky $x_1 \in A_1$ a $x_2 \in A_2$ tak, že $x \in \{x_1\}^{\perp\perp} \vee \{x_2\}^{\perp\perp}$.

(iv) Je-li $A \in S$, $x \in \Omega$, $x \notin A$, $x \notin A^{\perp}$, pak $A \cap (A^{\perp} \vee \{x\}^{\perp\perp})$ a $A^{\perp} \cap (A \vee \{x\}^{\perp\perp})$ jsou atomy v $\mathscr{S}$.

Резюме

ПРОЕКЦИИ И ПОКРЫТИЯ В МНОЖЕСТВЕ С ОРТОГОНАЛЬНОСТЬЮ

JAN HAVRDA

Рассматривается множество с отношением ортогональности $(\Omega, \perp)$ и порожденная им полная решетка с ортогональностью $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$. Если решетка $\mathscr{S}$ ортомодулярна и если для каждого $x \in \Omega$, $x \notin A$, $\{x\}^{\perp\perp}$—атом в $\mathscr{S}$, то следующие предложения эквивалентны

(i) Если $A \in S$, $x \in \Omega$, $x \notin A$, то $A \vee \{x\}^{\perp\perp}$ покрывает A.

(ii) Если $x \in \Omega$, $A \in S$, $x \notin A$, $x \notin A^{\perp}$, то существуют атомы $A_1 \subset A$, $A_2 \subset A^{\perp}$ такие, что $x \in A_1 \vee A_2$.

(iii) Если $A_1, A_2 \in S$, $A_1 \perp A_2$, $A_1 \neq \{o\} \neq A_2$, $x \in A_1 \vee A_2$, $x \notin A_1$, $x \notin A_2$, то существуют элементы $x_1 \in A_1$, $x_2 \in A_2$ такие, что $x \in \{x_1\}^{\perp\perp} \vee \{x_2\}^{\perp\perp}$.

(iv) Если $A \in S$, $x \in \Omega$, $x \notin A$, $x \notin A^{\perp}$, то $A \cap (A^{\perp} \vee \{x\}^{\perp\perp}$ и $A^{\perp} \cap (A \vee \{x\}^{\perp\perp})$—атомы в $\mathscr{S}$.

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