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A STUDY OF INDEPENDENCE IN A SET WITH ORTHOGONALITY

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Summary. We investigate a set with orthogonality \((\Omega, \bot)\) and the corresponding complete lattice with orthogonality \(\mathcal{S} = (S, \cap, \bot, \Omega, \{0\})\). We assume that the lattice \(\mathcal{S}\) is orthomodular and that it satisfies some natural assumptions. Let us suppose that \(0 \notin A \subset \Omega\) and that the set \(A\) contains at least two points. We then call the set \(A\) \(j\)-independent if \(\bigcap (A - \{x\})^{\perp} = \{0\}\), \(k\)-independent if \(B^{\perp} \cap C^{\perp} = \{0\}\) whenever \(A = B \cup C\), \(B \cap C = \emptyset\), \(B \neq \emptyset \neq C\), and \(l\)-independent if \(x \notin (A - \{x\})^{\perp}\) for all \(x \in A\). We call the set \(A\) \(I\)-independent if each finite subset of \(A\) which contains at least two points is \(i\)-independent for \((I, i) = (J, j)\), resp. \((I, i) = (K, k)\), resp. \((I, i) = (L, l)\). The article clarifies mutual relations of these concepts.

Keywords: set with orthogonality, orthomodular lattice, \(i\)-independent set, \(I\)-independent set.

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1. This paper carries on some ideas of [1] and presents three concepts of independent sets in a set with an orthogonality relation \((\Omega, \bot)\). It also pays attention to their interrelations. The motivation comes from the theory of linear spaces.

Let us recall that we call a relation \(\bot \subset \Omega \times \Omega\) an orthogonality relation if 1. \(\bot\) is symmetric, 2. there is a distinguished element \(0\) such that \(\{0\} \times \Omega \subset \bot\) and the intersection of \(\bot\) with the diagonal is exactly \((0, 0)\). The presence of an orthogonality relation on the set \(\Omega\) gives rise to a complete lattice \(\mathcal{S} = (S, \cap, \bot, \Omega, \{0\})\) where \(S\) consists of all subsets \(A\) of \(\Omega\) satisfying \(A = (A^{\perp})^{\perp}\). Here, \(\Omega\) plays the role of the unit element and \(\{0\}\) plays the role of the nought element.

Throughout the whole paper, we shall assume that the complete lattice \(\mathcal{S}\) is orthomodular and satisfies Axiom A and Axiom P:

**Axiom A.** For every \(x \in \Omega\), \(x \neq 0\), \(\{x\}^{\perp}\) is an atom in \(\mathcal{S}\).

**Axiom P.** If \(x \in \Omega\), \(A \subset S\), \(x \notin A\), \(x \notin A^{\perp}\), then there exist atoms \(A_1 \subset A\) and \(A_2 \subset A^{\perp}\) such that \(x \in A_1 \lor A_2\).

Let us restate some equivalent conditions on a lattice with an orthogonality relation \(\mathcal{S} = (P, \leq, \bot, 1, 0)\) which we shall use in the sequel.

1.1. \(\mathcal{S}\) is orthomodular.
1.2. If \(a, b \in P\), \(a \leq b\), then \(b = a \lor (a^{\perp} \land b)\).
1.3. If \(a, b \in P\), \(a \leq b\), \(a^{\perp} \land b = 0\), then \(a = b\).
1.4. If \(a, b, c \in P\), \(a \leq c\), \(b \leq c^{\perp}\), then \((a \lor b) \land c = a\).
2. Let \((\Omega, \perp)\) be a given set with an orthogonality relation. By the following definition we shall introduce three types of independence of subsets of \(\Omega\).

### 2.1. Definition

Let \(A\) be a subset of \(\Omega\) such that \(o \notin A\). Let us assume that the set \(A\) contains at least two points. We call the set \(A\) \(j\)-independent if and only if \(\bigcap_{x \in A} (A - \{x\})^{1\perp} = \{o\}\). We call the set \(A\) \(k\)-independent if and only if \(B^{1\perp} \cap C^{1\perp} = \{o\}\) whenever \(A = B \cup C, B \neq \emptyset \neq C, B \cap C = \emptyset\). We call the set \(A\) \(l\)-independent if and only if \(x \notin (A - \{x\})^{1\perp}\) for all \(x \in A\).

### 2.2. Lemma

1. Every \(j\)-independent set is \(k\)-independent.
2. Every \(k\)-independent set is \(l\)-independent.

**Proof.**

1. Let \(A\) be a \(j\)-independent set. We have \(\{o\} = \bigcap_{x \in A} (A - \{x\})^{1\perp} = \bigcap_{x \in B} (B \cup C - \{x\})^{1\perp} \cap \bigcap_{x \in C} (B \cup C - \{x\})^{1\perp} = C^{1\perp} \cap B^{1\perp}\) whenever \(A = B \cup C, B \cap C = \emptyset\).

2. Let \(A\) be a \(k\)-independent set. It is true that \(\{o\} = (A - \{x\})^{1\perp} \cap \{x\}^{1\perp}\) for all \(x \in A\), hence \(x \notin (A - \{x\})^{1\perp}\).

### 2.3. Lemma

The following statements are equivalent: 1. The set \(A\) is \(j\)-independent. 2. For every element \(y \in A^{1\perp}\), \(y \neq o\), there is an element \(a_y \in A\) such that \(y \notin (A - \{a_y\})^{1\perp}\). In addition, we have \(A^{1\perp} = (A - \{a_y\})^{1\perp} \lor \{y\}^{1\perp}\) for the elements \(y\) and \(a_y\) from Statement 2.

**Proof.**

1. \(\Rightarrow\) 2. If such an element \(a_y\) does not exist, then \(y \in (A - \{x\})^{1\perp}\) for every \(x \in A\), hence \(y \in \bigcap_{x \in A} (A - \{x\})^{1\perp} = \{o\}\) according to above, contrary to our hypothesis \(y \neq o\).

2. \(\Rightarrow\) 1. If \(\bigcap_{x \in A} (A - \{x\})^{1\perp} = \{o\}\), then there is an element \(y \in \bigcap_{x \in A} (A - \{x\})^{1\perp}\), \(y \neq o\). Hence \(o \notin y \in (A - \{x\})^{1\perp}\) for all \(x \in A\) — a contradiction.

According to Theorem 2.10 of [2], we have \((A - \{a_y\})^{1\perp} < (A - \{a_y\})^{1\perp} \lor \{a_y\}^{1\perp} = A^{1\perp}\). Since \((A - \{a_y\})^{1\perp} < (A - \{a_y\})^{1\perp} \lor \{y\}^{1\perp} \subseteq A^{1\perp}\), we have either \((A - \{a_y\})^{1\perp} = (A - \{a_y\})^{1\perp} \lor \{y\}^{1\perp}\) or \((A - \{a_y\})^{1\perp} \lor \{y\}^{1\perp} = A^{1\perp}\). But the first identity is not valid because, in that case, we should have \(y \in (A - \{a_y\})^{1\perp}\), contrary to our hypothesis. Lemma is proved.

### 2.4. Lemma

The following statements are equivalent: 1. The set \(A\) is \(k\)-independent. 2. The identity \(B^{1\perp} \cap C^{1\perp} = \{o\}\) is valid for every subsets \(B, C \subset A, B \cap C = \emptyset, B \neq \emptyset \neq C\).

**Proof.**

1. \(\Rightarrow\) 2. We have \(B^{1\perp} \cap C^{1\perp} \subseteq B^{1\perp} \cap (A - B)^{1\perp} = \{o\}\).

2. \(\Rightarrow\) 1. It suffices to put \(C = A - B\).
2.5. Lemma. The following statements are equivalent: 1. The set $A$ is $l$-independent. 2. The inequality $B^{11} + A^{11}$ holds for every subset $B \subset A$, $\emptyset \neq B \neq A$.

Proof. 1 $\Rightarrow$ 2. Let us suppose $B \subset A$, $\emptyset \neq B \neq A$ and $B^{11} = A^{11}$. Then there is an element $x \in A$, $x \notin B$, hence $B \subset A - \{x\} \subset A$. It follows that $B^{11} \subset (A - \{x\})^{11} \subset A^{11}$ which implies $x \in A \subset A^{11} = (A - \{x\})^{11} - a$ contradiction.

2 $\Rightarrow$ 1. Putting $B = A - \{x\}$ we have $(A - \{x\})^{11} \subset A^{11}$ and $(A - \{x\})^{11} + \neq A^{11}$ for all $x \in A$. It is true that $\{o\} \neq (A - \{x\})^{11} \cap A^{11} = (A - \{x\})^{11} \cap \cap [(A - \{x\})^{11} \vee \{x\}^{11}]$ in accordance with Statement 1.3. If the set $A$ is not $l$-independent, then there is an element $x \in A$ such that $x \in (A - \{x\})^{11}$, hence $\{x\}^{11} \subset (A - \{x\})^{11}$. We get $\{o\} \neq (A - \{x\})^{11} \cap [(A - \{x\})^{11} \vee \{x\}^{11}] = (A - \{x\})^{11} \cap (A - \{x\})^{11} = \{o\} - a$ contradiction.

2.6. Lemma. If a set $A$ is $i$-independent, then its every subset, which contains at least two points, is also $i$-independent for $i = j, k, l$.

Proof. $i = j$. Let $B \subset A$ and let $B$ contain at least two points. Then $\{o\} = \cap x \notin A (A - \{x\})^{11} = \cap x \notin B [(A - B)^{11} \cup (B - \{x\})^{11}] \cap \cap x \notin A - B (B^{11} \cup (A - B - \{x\})^{11}) \supset \cap x \notin B (B - \{x\})^{11} \cap B^{11} = \cap x \notin B (B - \{x\})^{11}$.

$i = k, i = l$. Proof is obvious.

2.7. Lemma. If a set $A$ is $i$-independent and $a \in \Omega$, $a \neq o$, $a \perp A$, then $A \cup \{a\}$ is also an $i$-independent set for $i = j, k, l$.

Proof. $i = j$. We have $\cap x \notin A (A - \{x\})^{11} = \{o\}, a \perp A$, hence $\{a\}^{11} \perp A^{11}$. It is true that $\cap x \notin A (A \cup \{a\} - \{x\})^{11} = A^{11} \cap \cap x \notin A (a)^{11} \cup (A - \{x\})^{11}] = \cap A^{11} \cap \cap x \notin A ([a]^{11} \vee (A - \{x\})^{11}] = \cap (A - \{x\})^{11} = \{o\}$ where the last but one identity follows from Statement 1.4.

$i = k$. On the one hand we have $A^{11} \cap \{a\}^{11} = \{o\}$, and on the other, if we have $A = B \cup C, B \cap C = \emptyset, B \neq \emptyset \neq C$, then $A \cup \{a\} = (B \cup \{a\}) \cup C$ and $(B \cup \{a\}) \cap C = \emptyset$. It is evident that $\{a\}^{11} = ([a]^{11} \vee B^{11}) \cap ([a]^{11} \vee C^{11})$. In accordance with Statements 1.2 and 1.4, we get $([a]^{11} \vee B^{11}) \cap ([a]^{11} \vee C^{11}) = ([a]^{11} \cap ([a]^{11} \vee B^{11}) \cap ([a]^{11} \vee C^{11})) = [a]^{11} \cap ([a]^{11} \vee B^{11}) \cap ([a]^{11} \vee C^{11}) = [a]^{11} \cap (B^{11} \cap C^{11}) = [a]^{11}$. Thus, $\{a\}^{11} = ([a]^{11} \vee B^{11}) \cap ([a]^{11} \vee C^{11}) = ([a]^{11} \vee B^{11}) \cap C^{11}$. Since the equality $\{a\}^{11} = ([a]^{11} \vee B^{11}) \cap C^{11} \cap C^{11}$ cannot hold, we see that $([a]^{11} \vee B^{11}) \cap C^{11} = \{o\}$. (Axiom A). This completes the proof.

$i = l$. We have $A^{11} = (A - \{x\})^{11} \cup ([A - \{x\})^{11} \cap A^{11}]$ for every $x \in A$ in view of Statement 1.2. Let $B_x$ be the set $(A - \{x\})^{11} \cap A^{11}$. Then $B_x \subset (A -$
\(- \{x\}\), hence we get \(A^{1 \perp} \cap B_{x}^{1} = [(A - \{x\})^{1 \perp} \vee B_{x}] \cap B_{x}^{1} = (A - \{x\})^{1 \perp}\) in virtue of Statement 1.4. If the set \(A\) is \(l\)-independent and if \(a \equiv o, a \perp A\), then \(\{a\}^{1 \perp} \subseteq A^{1} = (A - \{x\})^{1 \perp} \cap B_{x}^{1} \subseteq B_{x}^{1}\). We have \(\{a\}^{1 \perp} \vee (A - \{x\})^{1 \perp} = (A - \{x\})^{1 \perp} \vee (A^{1 \perp} \cap B_{x}^{1})\). Since \(\{a\}^{1 \perp} \subseteq ((\{a\}^{1 \perp} \vee A^{1 \perp}) \cap B_{x}^{1}\) and since \(A^{1 \perp} \subseteq \{a\}^{1 \perp}\), according to Statements 1.2 and 1.4 we get \((\{a\}^{1 \perp} \vee A^{1 \perp}) \cap B_{x}^{1} = \{a\}^{1 \perp} \vee \{a\}^{1 \perp} \cap (\{a\}^{1 \perp} \vee A^{1 \perp}) \cap B_{x}^{1}\) = \(\{a\}^{1 \perp} \vee (A^{1 \perp} \cap B_{x}^{1})\). Hence we have \(\{a\}^{1 \perp} \vee (A - \{x\})^{1 \perp} = ((\{a\}^{1 \perp} \vee A^{1 \perp}) \cap B_{x}^{1}\). If the relation \(x \in \{a\}^{1 \perp} \cap (A - \{x\})^{1 \perp}\) holds for some \(x \in A\), it follows that \(\{x\}^{1 \perp} = \{x\}^{1 \perp} \cap \{a\}^{1 \perp} \cap (A - \{x\})^{1 \perp}\) holds for some \(x \in A\), it follows that \(\{x\}^{1 \perp} = \{x\}^{1 \perp} \cap (\{a\}^{1 \perp} \vee A^{1 \perp}) \cap B_{x}^{1}\). Therefore \(\{x\}^{1 \perp} = \{x\}^{1 \perp} \cap A^{1 \perp} \cap B_{x}^{1} \cap A^{1 \perp} = (A - \{x\})^{1 \perp}\) - a contradiction.

**2.8. Lemma.** An \(i\)-independent set \(A \subseteq \Omega\) is a maximal \(i\)-independent set with respect to the set inclusion if and only if \(A^{1 \perp} = \Omega\) for \(i = j, k, l\).

**Proof.** 1. Let \(A\) be an \(i\)-independent set for \(i = j, k, l\) and let \(A^{1 \perp} \neq \Omega\). Hence \(A^{1} = \{o\}\) and there exists \(a \in A^{1}\), \(a \equiv o\). It is true that \(a \perp A\). The set \(A \cup \{a\}\) is also an \(i\)-independent set according to Lemma 2.7 for \(i = j, k, l\).

2. Let \(A\) be an \(i\)-independent set for \(i = j, k, l\) and let \(A^{1 \perp} = \Omega\). If \(A \subseteq B, A \neq B\), where the set \(B\) is also \(i\)-independent for \(i = j, k, l\), then the set \(A\) as well as the set \(B\) are \(l\)-independent in accordance with Lemma 2.2. It follows that \(\Omega = A^{1 \perp} \subseteq B^{1 \perp}\), \(A^{1 \perp} \neq B^{1 \perp}\) according to Lemma 2.5. However, this contradicts our hypothesis. Hence \(A\) is a maximal \(i\)-independent set for \(i = j, k, l\).

**2.9. Theorem.** Every \(i\)-independent set \(A \subseteq \Omega\) is a subset of maximal \(i\)-independent set for \(i = j, k, l\).

**Proof.** First, we shall prove the following statement: If \(A \subseteq S, A \neq \{o\}\), then \(A = \bigvee_{i \in I} \{a_{i}\}^{1 \perp}\) where \(a_{i} \equiv o, a_{i} \perp A_{j}\) for \(i \neq j, i, j \in I\). Indeed, let \(\{C_{k}: k \in K\}\) be a chain of orthogonal sets (i.e. when \(x, y \in C_{k}, x \equiv y, \) then \(x \perp y)\) such that \(C_{k}^{1 \perp} \subseteq A\) for all \(k \in K\). Hence \(\bigcup_{k \in K} C_{k} = C\) is an orthogonal set. Moreover, \((\bigcup_{k \in K} C_{k})^{1 \perp} = \bigvee_{k \in K} C_{k}^{1 \perp} \subseteq A\). It follows that there are maximal orthogonal sets \(D \subseteq A\). Then \(D^{1 \perp} = A\). If not, then \(D^{1 \perp} \subseteq A, D^{1 \perp} \neq \{o\}\). Hence \(D^{1} \cap A \neq \{o\}\) in virtue of Statement 1.3. Consequently, there is \(a \in D^{1} \cap A \neq \{o\}\). We have \((D \cup \{a\})^{1 \perp} \subseteq A\) and the set \(D \cup \{a\}\) is an orthogonal set, therefore the set \(D\) is not maximal. Our assertion is proved.

Now, let \(A\) be an \(i\)-independent set. If \(A^{1 \perp} = \Omega\), then \(A\) is a maximal \(i\)-independent set in view of Lemma 2.8 for \(i = j, k, l\). If \(A^{1 \perp} \neq \Omega\), then in accordance with our assertion above, \(A^{1} = \bigvee_{g \in I} \{a_{g}\}^{1 \perp}\) where \(a_{g} \equiv o, a_{g} \perp a_{h}\) for \(g \neq h, g, h \in I\). Let \(B\) stand for the set \(\{a_{h}: h \in I\}\). The set \(A \cup B\) has the property \((A \cup B)^{1 \perp} = A^{1 \perp} \vee \bigvee_{h \in I} \{a_{h}\}^{1 \perp} = A^{1 \perp} \vee A^{1} = \Omega\). We shall prove that \(A \cup B\) is an \(i\)-independent set for \(i = j, k, l\).
The following theorem is a generalization of Lemma 2.7.
2.10. Theorem. If the set \( A \) is i-independent and if \( a \notin A^{11} \), then \( A \cup \{a\} \) is also an i-independent set for \( i = j, k, l \).

Proof. \( i = j \). We have
\[
\bigcap \{(A \cup \{a\} - \{x\})^{11}\} = A^{11} \cap \bigcap \{[a]^{11} \lor (A - \{x\})^{11}\} \cap \bigcap \{[B_x \lor (A - \{x\})^{11}]\} \text{ where } A_x = (A - \{x\})^{11} \cap A^{11}\text{ and } B_x = (A - \{x\})^{11} \cap \bigcap \{[a]^{11} \lor (A - \{x\})^{11}\}.\]
Evidently, \( A_x \perp (A - \{x\})^{11} \) and \( B_x \perp (A - \{x\})^{11} \). Applying Statement 1.4 we get \( (A - \{x\})^{11} \lor [(A - \{x\})^{11} \cap A_x^{11}] = A_x^{11} \) and \( (A - \{x\})^{11} \lor [(A - \{x\})^{11} \cap B_x^{11}] = B_x^{11} \). Applying again Statement 1.4 we have \( (A_x^{11} \lor B_x^{11}) \cap (A - \{x\})^{11} = (A - \{x\})^{11} \lor [(A - \{x\})^{11} \cap A_x^{11}] \lor [(A - \{x\})^{11} \cap B_x^{11}] \cap (A - \{x\})^{11} \lor [(A - \{x\})^{11} \cap A_x^{11}] \lor [(A - \{x\})^{11} \cap B_x^{11}] \). Therefore,
\[
\bigcap \{A_x \lor B_x \lor (A - \{x\})^{11} \}= \bigcap \{[A_x \lor B_x \lor (A - \{x\})^{11}] \cap (A - \{x\})^{11}\} \text{ in accordance with Theorem 2.10 of [2]. }\]
\( A_x \) and \( B_x \) are atoms in the lattice \( \mathcal{S} \). If \( A_x = B_x \), then \( a \in \{A^{11} \lor (A - \{x\})^{11}\} = B_x \lor (A - \{x\})^{11} \lor A_x \lor (A - \{x\})^{11} = \{x\}^{11} \lor \lor (A - \{x\})^{11} = A^{11} \) - a contradiction. Thus, \( A_x \cap B_x = \{0\} \) and we have
\[
\bigcap \{(A \cup \{a\} - \{x\})^{11}\} = \bigcap \{(A - \{x\})^{11}\} = \{0\}.\]

\( i = k \). Let \( A = B \cup C \), \( B \cap C = \emptyset \), \( B \neq \emptyset \neq C \). Using Statement 1.2 we have
\[
(B^{11} \lor \{a\}^{11}) \cap C^{11} \subset (B^{11} \lor \{a\}^{11}) \cap (B^{11} \lor C^{11}) = (B^{11} \lor B_a) \cap (B^{11} \lor C_b) \text{ where } B_a = B^{11} \lor \{a\}^{11} \text{ and } C_b = B^{11} \lor C^{11}.\]
Since \( B_a \perp B^{11} \) and \( C_b \perp B^{11} \), it is true that \( (B^{11} \lor B_a) \cap (B^{11} \lor C_b) = B^{11} \lor (B_a \cap C_b) \) which can be proved in a similar way as in the first part of this proof. According to Theorem 2.10 of [2], \( B_a \) is an atom. If \( B_a \cap C_b \neq \{0\} \), then \( B_a = B_a \cap C_b = B_a \cap A^{11} \). It follows that \( a \in B^{11} \lor \{a\}^{11} = B^{11} \lor B_a = B^{11} \lor (B_a \cap A^{11}) \subset B^{11} \lor A^{11} \) - a contradiction. Therefore \( B_a \cap C_b = \{0\} \), hence \( (B^{11} \lor \{a\}^{11}) \cap C^{11} \subset B^{11} \lor C^{11} \). Since \( (B^{11} \lor \{a\}^{11}) \cap C^{11} \subset C^{11} \) we have \( (B^{11} \lor \{a\}^{11}) \cap C^{11} \subset B^{11} \cap C^{11} = \{0\} \).

\( i = k \). Proof of the statement coincides with the proof of Theorem 2.10 of [1].

The theorem is proved.

Let us introduce the following axiom.

**Axiom I.** If \( A \subseteq \Omega \) is an i-independent set, \( A = B \cup C \), \( B \neq \emptyset \neq C \), \( B \cap C = \emptyset \), then
\[
\bigcap \{A - \{x\}\}^{11} = B^{11}.\]

In accordance with Theorem 2.12 of [1], Axiom I is satisfied when \( C \) is a finite set. Let us now suppose that \( A \) is an orthogonal set. Then
\[
\bigcap \{B \cup C - \{x\}\}^{11} = \bigcap \{B^{11} \lor (C - \{x\})^{11}\} = B^{11} \lor \bigcap \{C - \{x\}\}^{11} \text{ because } B^{11} \cap [B^{11} \lor \lor (C - \{x\})^{11}] = (C - \{x\})^{11} \text{ for all } x \in C \text{ according to Statement 1.4. Hence
applying again Statement 1.4 we have $B^1 \cap \bigvee_{x \in C} (C - \{x\})^1 = B^1 \cap \bigvee_{x \in C} B^{1,1} \vee \bigvee_{x \in C} [B^1 \cap (C - \{x\})^1]$ $\bigvee_{x \in C} [B^1 \cap (C - \{x\})^1] = B^1 \cap \bigvee_{x \in C} B^{1,1} \vee \bigvee_{x \in C} (C - \{x\})^{1,1} = B^1 \cap \bigvee_{x \in C} B^{1,1} \vee \bigvee_{x \in C} (C - \{x\})^{1,1} = \bigcap_{x \in C} [B^{1,1} \vee (C - \{x\})^{1,1}]$. Further, we have $(C - \{x\})^{1,1} = C^{1,1} \cap \{x\}^{1,1}$ as a consequence of Statement 1.4 which implies $\bigcap_{x \in C} (C - \{x\})^{1,1} = \bigcap_{x \in C} (C^{1,1} \cap \{x\}^{1,1}) = C^{1,1} \cap \{x\}^{1,1} = C^{1,1} \cap C^1 = \{\emptyset\}$ Therefore we have $\bigcap_{x \in C} (B \cup C - \{x\})^{1,1} = B^{1,1}$. Thus we see that Axiom I is also satisfied when $A$ is an orthogonal set.

2.11. Lemma. If $A$ is an $l$-independent set and if the lattice $\mathcal{L}$ satisfies Axiom I then $A$ is also $j$-independent.

Proof. For an element $a \in A$ we have $\bigcap_{x \in C} (A - \{x\})^{1,1} = (A - \{a\})^{1,1} \cap \bigcap_{x \in C} (A - \{x\})^{1,1} = (A - \{a\})^{1,1} \cap \{a\}^{1,1} = \{\emptyset\}$.

2.12. Definition. Let $\emptyset \neq A \subset \Omega$, $\emptyset \neq A$ and let us assume that the set $A$ contains at least two points. We shall say that the set $A$ is $J$-independent if and only if its every finite subset which contains at least two points is $j$-independent. We shall say that the set $A$ is $K$-independent if and only if its every finite subset which contains at least two points is $k$-independent. We shall say that the set $A$ is $L$-independent if and only if its every finite subset which contains at least two points is $l$-independent.

2.13. Theorem. Let $\emptyset \neq A \subset \Omega$, $\emptyset \neq A$ and let us suppose that the set $A$ contains at least two points. Then the following statements are equivalent. 1. The set $A$ is $J$-independent. 2. The set $A$ is $K$-independent. 3. The set $A$ is $L$-independent.

Proof follows from Lemma 2.2, Theorem 2.12 of [1] and Lemma 2.11.

Let us note that, according to Lemma 2.6, every $j$-independent set is $J$-independent, every $k$-independent set is $K$-independent and every $l$-independent set is $L$-independent.

Let $\emptyset \neq a \in \Omega$ and let the set $A = \{a\}^{1,1} - \{\emptyset\}$ contain at least two points. Then for all $x \in A$ we have $(A - \{x\})^{1,1} = \{a\}^{1,1}$. If $A = B \cup C$, $B \cap C = \emptyset$, $B \neq \emptyset \neq C$, hence $\bigcap_{x \in C} (B \cup C - \{x\})^{1,1} = \{a\}^{1,1} = B^{1,1}$. This example shows that the assertion of Axiom I may be satisfied even when the set $A$ is not $l$-independent.

Literature

Uvažuje se množina s ortogonalitou \( (\Omega, \perp) \) a ji odpovídající úplný svaz s ortogonalitou \( S = (\mathcal{S}, \subset, \perp, \Omega, \{o\}) \). Předpokládá se, že svaz \( S \) je ortomodulární a splňuje některé další předpoklady. Nechť \( o \notin A \subset \Omega \), \( A \) obsahuje alespoň dva prvky. Podmnožina \( A \) se nazývá \( j \)-nezávislá, když \( \bigcap (A - \{x\})_{\perp} = \{o\} \), nazývá se \( k \)-nezávislá, když \( B_{\perp} \cap C_{\perp} = \{o\} \), kdykoliv \( A = \bigcup_{x \in A} B \neq C \), \( B \cap C = 0 \), nazývá se \( l \)-nezávislá, když \( x \notin (A - \{x\})_{\perp} \) pro všechna \( x \in A \). Podmnožina \( A \) se nazývá \( \ell \)-nezávislá, když každá její konečná podmnožina, která obsahuje alespoň dva prvky, je \( i \)-nezávislá, kde \( (I, i) = (J, j), (K, k), (L, l) \). Článek se zabývá vlastnostmi těchto pojmů a vztahy mezi nimi.

Resumé

ИЗУЧЕНИЕ НЕЗАВИСИМОСТИ В МНОЖЕСТВЕ С ОРТОГНАЛЬНОСТЬЮ

Jan Havrda

Рассматривается множество с отношением ортогональности \( (\Omega, \perp) \) и порожденная им полная решетка с ортогональностью \( \mathcal{S} = (\mathcal{S}, \subset, \perp, \Omega, \{o\}) \). Предполагается, что решетка \( \mathcal{S} \) ортомодульна и удовлетворяет некоторым дальнейшим предположениям. Пусть \( o \notin A \subset \Omega \), где в А по крайней мере два элемента. Множество \( A \) называется \( j \)-независимым, если \( \bigcap (A - \{x\})_{\perp} = \{o\} \); \( k \)-независимым, если \( B_{\perp} \cap C_{\perp} = \{o\} \), как только \( A = B \cup C \), \( B \cap C = 0 \); \( l \)-независимым, если \( x \notin (A - \{x\})_{\perp} \) для всех \( x \in A \). Множество \( A \) называется \( \ell \)-независимым, если каждое конечное подмножество, в котором по крайней мере два элемента, \( i \)-независимо, \( (I, i) = (J, j), (K, k), (L, l) \). Статья занимается взаимными отношениями между этими понятиями.

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