A NOTE ON THE COMPUTATIONAL COMPLEXITY OF COMPUTING THE EDGE ROTATION DISTANCE BETWEEN GRAPHS

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Summary. The problem of computing the edge rotation distance between trees is shown to be NP-hard.

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INTRODUCTION

In [1] Chartrand, Saba and Zou introduced the concept of the edge rotation distance between isomorphism classes of graphs.

We say that a graph G can be transformed into a graph H by an edge rotation if G contains distinct vertices u, v and w such that $uv \in E(G)$, $uw \notin E(G)$ and $H = G - uv + uw$.

The *edge rotation distance* between graphs G and H, written erd (G, H), is defined as the minimum number of edge rotations needed to transform G into a graph isomorphic to H. In [1] it was shown that on the family of graphs having a fixed order and size, the distance function erd produces a metric space. Further, an upper bound for erd was presented. On the other hand, one may ask if there is a polynomial algorithm that computes erd. (Our *NP*-completeness terminology is that of [2].) Since

$$\operatorname{erd}(G, H) = 0 \Leftrightarrow G \cong H$$

this question is also interesting from the *NP*-theoretical point of view. The affirmative answer would solve the open problem about *NP*-completeness of testing two graphs for isomorphism.

In this paper the above question is answered in negative by showing the following problem *ERD* to be *NP*-hard:

ERD : *INSTANCE*: Two graphs G, H having the same finite order and the same size, positive integer k;

QUESTION: Is erd $(G, H) \leq k$?

RESULT

In this section we will show that the problem *ERD* is *NP*-complete even when restricted to trees. This will be done by transforming the 3-*PARTITION* problem to *ERD*. The 3-*PARTITION* problem is known to be *NP*-complete in the strong sense [2, p. 224] and is introduced as follows:

INSTANCE: Set A of $3m$ elements, a bound $B \in \mathbb{Z}^+$ (set of all positive integers) and a size $s(a) \in \mathbb{Z}^+$ for each $a \in A$ such that $B/4 < s(a) < B/2$ and such that $\sum_{a \in A} s(a) = Bm$;

QUESTION: Can A be partitioned into m disjoint sets $A_1, \ldots, A_m$ such that for $i = 1, \ldots, m$, $\sum_{a \in A_i} s(a) = B$?

Note that each A_i must contain exactly three elements from A.

Theorem. *The problem ERD is NP-complete even when restricted to trees.*

Proof. As is customary with the proofs of *NP*-completeness we omit the trivial verification that *ERD* (when restricted to trees) is in the class *NP*.

Now, given an instance of 3-*PARTITION* $A = \{a_1, \ldots, a_{3m}\}$, $B \in \mathbb{Z}^+$, and $s(a_1), \ldots, s(a_{3m})$ in $\mathbb{Z}^+$, the corresponding instance of *ERD* is constructed by the following procedure:

Let $P_1, \ldots, P_{3m}$ be $3m$ distinct paths such that each P_i has $s(a_i)$ vertices, and let e_i, f_i denote the two endpoints of path P_i. The graph G consists of paths $P_1, \ldots, P_{3m}$,

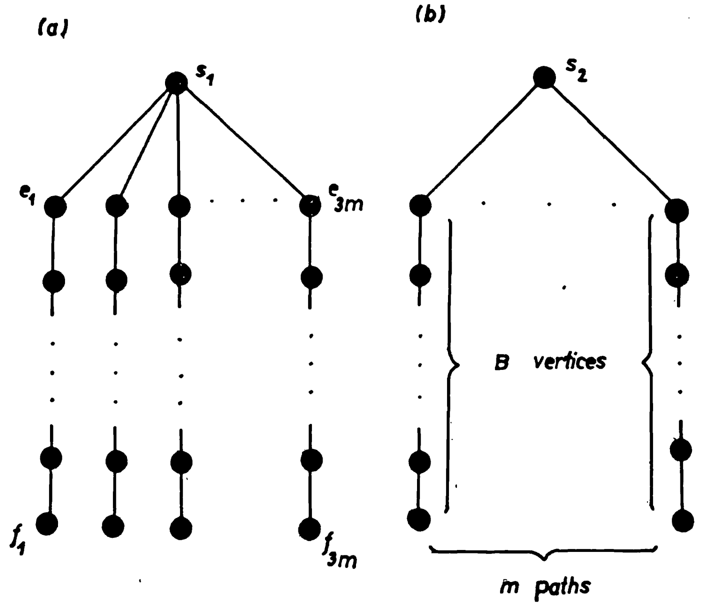

Fig. 1. (a) Graph G (b) Graph H

all attached by their endpoints $e_1, \ldots, e_{3m}$ to an additional common vertex s_1. The graph H consists of m paths of B vertices each, all joined at the end to a new common vertex s_2. Observe that both G and H are trees of the same order $mB + 1$ and the same size mB, see Figure 1.

Finally, we put $k = 2m$.

Let us suppose that *ERD* has "yes"-solution. Since the maximum degree of vertices in H is m while the vertex s_1 is of degree $3m$ in G, at least $2m$ edge rotations are needed to transform G into a graph isomorphic to H. Therefore there exists a sequence

$$G \cong G_1, \ldots, G_{2m+1} \cong H$$

of graphs such that

$$G_{i+1} = G_i - e_k s_1 + e_k f_l \quad (1 \leqq k \neq l \leqq 3m).$$

Consequently, we have

$$\sum_{a \in A_i} s(a) = B \quad (1 \leqq i \leqq m) \quad \text{where}$$

$A_i = \{a_{i_1}, a_{i_2}, a_{i_3}\} \Leftrightarrow e_{i_1}, e_{i_2}, e_{i_3}$ lie on the same path in the graph $H - s_2$ $(1 \leqq i_1 \neq$ $\neq i_2 \neq i_3 \neq i_1 \leqq 3m)$, and the 3-*PARTITION* problem has "yes"-solution.

The converse is also true. Given a solution to the 3-*PARTITION* problem, suitable $2m$ edge rotations can be found by "rotating" edges $e_{i_2}s_1(e_{i_3}s_1)$ to $e_{i_2}f_{i_1}(e_{i_3}f_{i_2}$, respectively) in G according to the partitioned sets $A_i = \{a_{i_1}, a_{i_2}, a_{i_3}\}$.

Since the graphs G, H have the number of vertices of the order of the numbers involved in 3-*PARTITION*, rather than the number of bits required to represent those numbers only the pseudopolynomial transformation from 3-*PARTITION* to *ERD* was exhibited. This does not matter, however, since 3-*PARTITION* is *NP*-complete in the strong sense. Hence *ERD* is *NP*-complete even when restricted to trees, *QED*.

References

[1] *G. Chartrand, F. Saba, H. Zou:* Edge rotations and distance between graphs. Časopis pěst. mat. *100* (1975), 371–373.

[2] *M. R. Garey, D. S. Johnson:* Computers and Intractability: a guide to the theory of *NP*-copleteness. Freeman, San Francisco, 1979.

Souhrn

POZNÁMKA O VÝPOČETNÍ SLOŽITOSTI VÝPOČTU HRANOVĚ ROTAČNÍ VZDÁLENOSTI MEZI GRAFY

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Ukazuje se, že problém určení hranově rotační vzdálenosti mezi dvěma stromy je *NP*-obtížný

Резюме

ЗАМЕЧАНИЕ О ВЫЧИСЛИТЕЛЬНОЙ СЛОЖНССТИ ВЫЧИСЛЕНИЯ РЕБЕРНО-ВРАЩАТЕЛЬНОГО РАССТОЯНИЯ МЕЖДУ ДВУМЯ ГРАФАМИ

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Доказывается, что проблема вычисления реберно-вращательного расстояния между двумы деревьями NP-трудная.

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