Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 113 (1988), No. 4, 337--341

Persistent URL: `http://dml.cz/dmlcz/118346`

113 (1988) ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY No. 4, 337—341

A SIMPLE PROOF OF THE RADEMACHER THEOREM

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(Received September 7, 1984)

Summary. A simple proof of the Rademacher theorem is given.

Keywords: Rademacher theorem, Lipschitz functions, differentiability.

AMS Classification: 26B05.

1. PRELIMINARIES

The well-known Rademacher theorem can be stated in the following form.

Theorem R. *Any Lipschitz function on R^n is almost everywhere Frechet differentiable.*

Rademacher [2] proved in fact a more general theorem and this is the reason why his proof is rather complicated. Stepanov [4] showed that Rademacher's proof gives that any function which is Lipschitz at any point of R^n is almost everywhere Frechet differentiable. Theorem R is not so deep and it admits essentially simpler proofs. A simple proof using distributions is contained in [1] (for the more detailed proof see [3]). The aim of the present note is to give an alternative more elementary simple proof. Note that the only relatively difficult point is the proof of the fact that the points at which a Lipschitz function has all directional derivatives but has not a Gateaux derivative form a null set. It is the proof of this fact in [1] that uses the distributions. We give a proof of this fact which is based on the Fubini theorem.

The one-dimensional Rademacher theorem is an easy consequence of well-known classical theorems. The usual argument uses the fact that any Lipschitz function on a bounded closed interval is absolutely continuous. The alternative proof uses the obvious fact that if K is a Lipschitz constant for f, then the function $x \mapsto f(x) + Kx$ is a nondecreasing function.

In what follows we shall use the one-dimensional Rademacher theorem, the Fubini theorem and the following simple facts on Lipschitz functions. We say that f is K-Lipschitz, if K is a Lipschitz constant for f.

(a) If $\mathscr{F}$ is a family of K-Lipschitz functions on $M \subset R^n$ and $s(x) = \sup\{f(x), f \in \mathscr{F}\}$ is finite on M, then s is K-Lipschitz.

(b) If (f_n) is a sequence of K-Lipschitz functions on $M \subset R^n$, $\lim_{n\to\infty} f_n(x) = f(x) \in R$ for $x \in M$, then f is K-Lipschitz on M.

(c) If f is K-Lipschitz on R^n, $a \in R^n$, $c, t \in R$, $t \neq 0$, then the functions $x \mapsto f(x) + c$, $x \mapsto f(a + x)$, $x \mapsto f(tx)/t$ are K-Lipschitz on R^n.

(d) Let $M \subset R^n$ be compact and let f_τ for any $\tau > 0$ be a K-Lipschitz function on M such that $f_{\tau_1}(x) \leqq f_{\tau_2}(x)$ for $\tau_1 \leqq \tau_2$ and $x \in M$. If $\lim_{\tau\to 0+} f_\tau(x) = g(x) \in R$, then g is a K-Lipschitz function on M and the convergence is uniform.

The statements (a), (b), (c) are well-known and easy to prove and (d) is an immediate consequence of (b) and Dini's theorem on the monotone convergence of continuous functions.

If f is a function on R^n, $x, v \in R^n$, then we define the directional derivative

$$D_v f(x) = \lim_{t\to 0} (f(x + vt) - f(x))/t \tag{1}$$

and the "directional Dini derivatives"

$$\bar{D}_v f(x) = \overline{\lim_{t\to 0}}\, (f(x + vt) - f(x))/t\,,$$
$$D_v^* f(x) = \underline{\lim_{t\to 0}}\, (f(x + vt) - f(x))/t\,.$$

It is easy to see that $D_v f(x)$ exists iff $\bar{D}_v f(x) = D_v^* f(x)$. The function f is said to be Gateaux differentiable at x if for any $v \in R^n$, $D_v f(x)$ exists and the function $v \mapsto D_v f(x)$ is linear. It is a well-known easy fact that f is Frechet differentiable at x iff it is Gateaux differentiable at x and the limit (1) is uniform on the unit sphere $S := \{v;\ \|v\| = 1\}$.

2. Proof. It is natural to split the proof of the Rademacher theorem in R^n $(n > 1)$ into two independent parts. Namely, it is sufficient to prove the following propositions.

Proposition 1. *Let a Lipschitz function f on R^n be Gateaux differentiable at a point $x \in R^n$. Then it is Frechet differentiable at x.*

Proposition 2. *Any Lipschitz function f on R^n is Gateaux differentiable almost everywhere.*

Proof of Proposition 1. For any $\tau > 0$ put

$$\bar{g}_\tau^x(v) = \bar{g}_\tau(v) = \sup\{(f(x + tv) - f(x))/t;\ t \in \langle -\tau, \tau\rangle \setminus \{0\}\}\,,$$
$$g_\tau^x(v) = g_\tau(v) = \inf\{(f(x + tv) - f(x))/t;\ t \in \langle -\tau, \tau\rangle \setminus \{0\}\}\,.$$

For any $t \in \langle -\tau, \tau\rangle \setminus \{0\}$ the function $v \mapsto (f(x + tv) - f(x))/t$ is K-Lipschitz by (c). Since $|f(x + tv) - f(x)/t| \leqq K\|v\|$, the functions $\bar{g}_\tau(v)$, $g_\tau(v)$ are K-Lipschitz on R^n by (a). For $0 < \tau < \tau_1$, clearly

(2) $$g_{\tau_1}(v) \leqq g_\tau(v) \leqq (f(x + \tau v) - f(x))/\tau \leqq \bar{g}_\tau(v) \leqq \bar{g}_{\tau_1}(v)$$

and

(3) $$\lim_{\tau\to 0+} \bar{g}_\tau(v) = \bar{D}_v f(x)\,, \quad \lim_{\tau\to 0+} g_\tau(v) = D_v^* f(x)\,.$$

The limits (3) are uniform on the unit sphere S by (d). Since $\bar{D}_v f(x) = D_v^* f(x) = D_v f(x)$ we obtain by (2) that the limit (1) is uniform on S and therefore f is Frechet differentiable at x.

For the proof of Proposition 2 we need the following measurability lemma.

Lemma. *Let f be a Lipschitz function on R^n, $\tau > 0$, $v \in R^n$. Then the functions*

$$x \mapsto \bar{g}_\tau^x(v)\,, \quad x \mapsto g_\tau^x(v)\,, \quad x \mapsto \bar{D}_v(x)\,, \quad x \mapsto D_v^*(x)$$

are Lebesgue measurable on R^n.

Proof. We have

$$\bar{g}_\tau^x(v) = \sup\{(f(x + tv) - f(x))/t;\ t \in \langle -\tau, \tau\rangle \setminus \{0\}\} =$$
$$= \sup\{(f(x + tv) - f(x))/t;\ t \in \langle -\tau, \tau\rangle \setminus \{0\},\ t \text{ rational}\}\,.$$

Since for any t the function $x \mapsto (f(x + tv) - f(x))/t$ is continuous, we obtain that $x \mapsto \bar{g}_\tau^x(v)$ is measurable. Since $\bar{D}_v(x) = \lim_{n\to\infty} \bar{g}_{1/n}^x(v)$, the function $x \mapsto \bar{D}_v(x)$ is measurable as well. The proofs for $g_\tau^x(v)$ and $D_v^*(x)$ are quite analogous.

Proof of Proposition 2. Denote by G the set of all $x \in R^n$ at which f is not Gateaux differentiable, by A^v the set of all $x \in R^n$ at which $D_v f(x)$ does not exist and put $A = \bigcup\{A^v;\ v \in R^n\}$. Let C be a countable dense subset of R^n. Then we have $A = \bigcup\{A^v;\ v \in C\}$. In fact, if $x \notin \bigcup_{v\in C} A^v$, then $\bar{D}_v f(x) = D_v^* f(x)$ for $v \in C$ and since the functions $v \mapsto \bar{D}_v f(x)$, $v \mapsto D_v^* f(x)$ are Lipschitz by (3) and (b), we have $\bar{D}_v f(x) = D_v^* f(x)$ for any $v \in R^n$ and therefore $x \notin A$. Since $A^v = \{x;\ \bar{D}_v f(x) \neq D_v^* f(x)\}$, we have by Lemma that A^v is measurable for any $v \in R^n$. Since for any line 1 parallel to v the linear measure of $A^v \cap 1$ is zero by the one-dimensional Rademacher theorem, we obtain by the Fubini theorem that A^v is a null set. Consequently, A is a null set. It remains to prove that $G - A$ is a null set. Let $x \in G - A$ be given. Then the function $v \mapsto D_v f(x)$ is defined but not linear on R^n. Since it is clearly homogeneous, there exist $v_1, v_2 \in R^n$ such that $D_{v_1} f(x) + D_{v_2} f(x) - D_{v_1+v_2} f(x) \neq 0$. From the continuity of the function $v \mapsto D_v f(x)$ we easily obtain that there exist $w_1, w_2 \in C$ such that $D_{w_1} f(x) + D_{w_2} f(x) - D_{w_1+w_2} f(x) \neq 0$. Thus if we denote for $r_1, r_2 \in R$ and $v_1, v_2 \in R^n$

$$B(v_1, v_2, r_1, r_2) =$$
$$= \{x \notin A;\ D_{v_1} f(x) > r_1,\ D_{v_2} f(x) > r_2,\ D_{v_1+v_2} f(x) < r_1 + r_2\}\,,$$

$$B^*(v_1, v_2, r_1, r_2) =$$
$$= \{x \notin A;\ D_{v_1} f(x) < r_1,\ D_{v_2} f(x) < r_2,\ D_{v_1+v_2} f(x) > r_1 + r_2\},$$

we easily see that

$$G - A \subset \bigcup\{B(v_1, v_2, r_1, r_2) \cup B^*(v_1, v_2, r_1, r_2);$$
$$v_1, v_2 \in C,\ r_1, r_2 \text{ rational}\}.$$

We easily see that it is sufficient to prove that $B(v_1, v_2, r_1, r_2)$ is a null set for fixed v_1, v_2, r_1, r_2. Clearly

$$B(v_1, v_2, r_1, r_2) \subset \bigcup_{m=1}^{\infty} B(v_1, v_2, r_1, r_2, m),$$

where $B(v_1, v_2, r_1, r_2, m)$ is the set of all points $x \in R^n$ for which

(4) $$g^x_{1/m}(v_1) = \inf\{(f(x + tv_1) - f(x))/t;\ 0 < |t| \leqq 1/m\} > r_1,$$

(5) $$g^x_{1/m}(v_2) = \inf\{(f(x + tv_2) - f(x))/t;\ 0 < |t| \leqq 1/m\} > r_2,$$

(6) $$\bar{g}^x_{1/m}(v_1 + v_2) = \sup\{(f(x + tv_1 + tv_2) - f(x))/t;$$
$$0 < |t| \leqq 1/m\} < r_1 + r_2.$$

Let an index m be fixed. By Lemma the set $B(v_1, v_2, r_1, r_2, m)$ is measurable and therefore by the Fubini theorem it is sufficient to prove that the linear measure of the set $T := B(v_1, v_2, r_1, r_2, m) \cap 1$ is zero for any line 1 parallel to $v_1 + v_2$ (we can suppose $v_1 + v_2 \neq 0$, since in the opposite case $B(v_1, v_2, r_1, r_2) = \emptyset$). We shall prove that T is even countable. For this purpose it is sufficient to show that for any $x, y \in T$, $x \neq y$, the inequality $\|x - y\| \geqq \|v_1 + v_2\|/m$ holds. Suppose on the contrary that points $x \neq y$ for which $\|x - y\| < \|v_1 + v_2\|/m$ are given. We can suppose that $y = x + t(v_1 + v_2)$, $0 < t < 1/m$. Since $x \in B(v_1, v_2, r_1, r_2, m)$, we obtain $(f(x + tv_1) - f(x))/t > r_1$ by (4) and $(f(y) - f(x))/t = (f(x + tv_1 + tv_2) - f(x))/t < r_1 + r_2$ by (6). Since $y \in B(v_1, v_2, r_1, r_2, m)$, we have $-(f(x + tv_1) - f(y))/t = -(f(y - tv_2) - f(y))/t > r_2$ by (5). These inequalities imply

$$f(x + tv_1) - f(x) > r_1 t,$$
$$f(y) - f(x) < r_1 t + r_2 t,$$
$$f(x + tv_1) - f(y) < -r_2 t,$$

which yields a contradiction.

References

[1] *C. B. Morrey:* Multiple Integrals in the Calculus of Variations. Berlin, 1966.

[2] *H. Rademacher:* Über partielle und totale Differenzierbarkeit I. Math. Ann. *89* (1919), 340—359.

[3] *J. Saint-Pierre:* Sur le théorème de Rademacher, Seminaire d'analyse convexe, Montpellier 1982, Exposé no. 2, 2.1—2.10.
[4] *H. Stepanoff:* Über totale Differenzierbarkeit, Math. Ann. *90* (1923), 318—320.
[5] *H. Federer:* Geometric measure theory. Springer-Verlag, 1969, 212—217.

Souhrn

JEDNODUCHÝ DŮKAZ RADEMACHEROVY VĚTY

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V článku je podán jednoduchý důkaz Rademacherovy věty.

Резюме

ПРОСТОЕ ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ РАДЕМАХЕРА

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Статья содержит простое доказательство теоремы Радемахера.

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