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Časopis pro pěstování matematiky, Vol. 113 (1988), No. 4, 435--436

Persistent URL: http://dml.cz/dmlcz/118349
A NOTE ON DETERMINACY OF MEASURES

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(Received October 23, 1986)

Summary. In the article it is shown that the Cramér-Wold theorem implies a stronger form of the Christensen theorem.

Keywords: determining set, probability measure.

AMS Classification: 28A05.

Let \( \mathcal{B}(\mathbb{R}^n) \) denote the collection of all Borel subsets of \( \mathbb{R}^n \) and let \( \mathcal{C} \) be a subset of \( \mathcal{B}(\mathbb{R}^n) \). Let \( \mathcal{C} \) be called determining when the following statement holds: If \( \mu_1, \mu_2 \) are two probability measures on \( \mathcal{B}(\mathbb{R}^n) \) which agree on \( \mathcal{C} \) then they are necessarily identical. The theorem of Christensen ([3]) says that the collection of all open balls is determining and the theorem of Cramér and Wold ([2]) says that the collection of all open half-spaces is determining. In this note we observe that the Cramér-Wold theorem implies a stronger form of the Christensen theorem. (As a by-product we obtain another proof of the Christensen theorem. For further discussion on the determinacy of measures, the reader is referred to [1], [4], [5], [6] and [7].)

Theorem. Let \( p \) be a point in \( \mathbb{R}^n \) \( (n \in \mathbb{N}) \) and let \( \mathcal{C} \) denote the collection of all open balls having \( p \) on the boundary. Then \( \mathcal{C} \) is determining.

Proof. Let \( \mu_1, \mu_2 \) agree on \( \mathcal{C} \). Applying a suitable transformation and multiple if necessary, we may assume that \( p = 0 \in \mathbb{R}^n \) and \( \mu_1\{0\} = \mu_2\{0\} = 0 \). Let \( \mathcal{C}_1 \) denote the collection of all open half-spaces which have 0 on the boundary. Put \( \mathcal{D} = \mathcal{C} \cup \mathcal{C}_1 \). Then \( \mu_1, \mu_2 \) agree on \( \mathcal{D} \). Indeed, each open half-space in \( \mathcal{C}_1 \) can be obtained as a union of an increasing sequence of balls in \( \mathcal{C} \). Hence \( \mu_1, \mu_2 \) have to agree on \( \mathcal{C}_1 \) in view of their monotone continuity.

Let now \( \varphi: \mathbb{R}^n \to \mathbb{R}^n \) be a mapping such that \( \varphi(0) = 0 \) and \( \varphi(x) = x/\|x\|^2 \) otherwise. Then \( \varphi \) is obviously a Borel isomorphism. One can easily show that \( \varphi(\mathcal{D}) \) is exactly the collection of all open half-spaces in \( \mathbb{R}^n \). By our assumption, the measures \( \mu_1 \varphi^{-1}, \mu_2 \varphi^{-1} \) agree on \( \varphi(\mathcal{D}) \) and therefore \( \mu_1 \varphi^{-1} = \mu_2 \varphi^{-1} \) (the Cramér-Wold theorem). This means that \( \mu_1 = \mu_2 \) and the proof is complete.
References


Souhrn
POZNÁMKA O URČENOSTI MĚR
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V článku je ukázáno, že Cramérova-Woldova věta implikuje silnější verzi Christensenovy věty.

Резюме
ЗАМЕЧАНИЕ ОБ ОПРЕДЕЛЕННОСТИ МЕР
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В работе показано, что теорема Крамера-Волда влечет за собой более сильный вариант теоремы Христенсена.

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436