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A NOTE ON WEAK HIDDEN VARIABLES

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Summary. We consider a σ-additive version of "centrally additive" hidden variables as introduced in [9]. As the main result we construct a logic without sufficiently many centrally additive dispersion free states. Consequently, this logic does not admit weak hidden variables.

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NOTIONS AND RESULTS

In the logico-algebraic approach to the foundations of quantum mechanics, the hidden variables hypothesis expresses by the presence of "sufficiently many" two-valued states (see [3], [5], [8], [11], etc.). Since many important logics have no two-valued states (see [1], [2], [7]), it is natural that generalized types of hidden variables have been considered ([6], [9]). In this note we introduce and shortly analyse one such generalization. Although the main result is in fact negative (it implies the absence of hidden variables), the investigation led us to a construction of a logic having rather special central properties.

Let us review the basic notions as we shall use them in the sequel. By a *logic* we mean a σ-orthomodular partially ordered set (see e.g. [3]). If L is a logic then by $C(L)$ we denote the set of all absolutely compatible elements of L (i.e. $C(L) = \{a \in L,\ a$ is compatible to each $b \in L\}$). The set $C(L)$, which is known to be a Boolean σ-algebra (in L), is called the centre of L.

We say that a mapping $h: L \to \{0, 1\}$ is a *central* $0-1$ *state* if

(i) $h(1) = 1$,

(ii) $h(a) + h(a') = 1$ for any $a \in L$,

(iii) $h(a) \leqq h(b)$ whenever $a, b \in L$ and $a \leqq b$,

(iv) $h(\bigvee_{i \in N} a_i) = \sum_{i \in N} h(a_i)$ whenever $a_i \in L$ $(i \in N)$, $a_i \leqq a_j'$ for any $i \neq j$ and at most one of a_i's does not belong to $C(L)$.

Of course, if L is Boolean the central $0-1$ states coincide with the $0-1$ states.

We have the following result:

Theorem 1. *Let L be a logic and let h be a (central) $0-1$ state on $C(L)$. Then there is a central $0-1$ state $\tilde{h}$ on L such that the restriction of $\tilde{h}$ to $C(L)$ is h.*

Proof. We aply the following result [9]. For the logic L there exists a Boolean algebra B and an injective mapping $\varphi: L \to B$ such that the following conditions are satisfied:

$\varphi(1) = 1$,
$\varphi(a') = \varphi(a)'$ for each $a \in L$,
$\varphi(a) \leqq \varphi(b)$ whenever $a, b \in L$, $a \leqq b$,
$\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ whenver $a, b \in L$, $a \leqq b'$, and $a \in C(L)$.

In particular, φ is a Boolean embedding of $C(L)$ into B. Now let h be a central $0-1$ state on $C(L)$. By the theorem of Horn and Tarski [4] h can be extended to a two-valued finitely additive measure on B. Denote this measure by k and put $\tilde{h}(a) = k(\varphi(a))$. We claim that $\tilde{h}$ is the required extension. Indeed, $\tilde{h}|_{C(L)} = h$ and if a_i is a sequence of mutually orthogonal elements of L and $a_i \in C(L)$ for $i > 1$, then $\tilde{h}(\bigvee_{i \in N} a_i) = k(\varphi(\bigvee_{i \in N} a_i)) = k(\varphi(a_1) \vee \varphi(\bigvee_{i>1} a_i)) = k(\varphi(a_1)) + k(\bigvee_{i>1} a_i) = \tilde{h}(a_1) + h(\bigvee_{i>1} a_i) = \sum_{i \in N} \tilde{h}(a_i)$. The proof is complete.

We say that L possesses weak hidden variables, if for any pair $a, b \in L$ with $a \nleqq b$ there is a central $0-1$ state $h: L \to \{0, 1\}$ such that $s(a) = 1$ and $s(b) = 0$. Similarly as in the finitely additive case we have the following characterization.

Proposition 2. *A logic L possesses weak hidden variables if and only if there is an injective mapping $\psi: L \to B$ into a Boolean σ-algebra B of subsets of a set such that*

(i) $\psi(1) = 1$,
(ii) $\psi(a) \leqq \psi(b)$ *if and only if* $a \leqq b$ $(a, b \in L)$,
(iii) $\psi(a') = \psi(a)'$ *for any* $a \in L$,
(iv) $\psi(\bigvee_{i \in N} a_i) = \bigvee_{i \in N} \psi(a_i)$ *whenever* $a_i \in L (i \in N)$, $a_i \leqq a_j'$ *for any* $i \neq j$ *and* $a_i \in C(L)$ *for* $i > 1$.

Proof. If $\psi: L \to B$ is a mapping with the properties (i)–(iv) and if $a \nleqq b$ then $\psi(a) \setminus \psi(b)$ is nonvoid. If we take a point $p \in \psi(a) \setminus \psi(b)$ and consider the state $s_p: B \to \{0, 1\}$ concentrated in $\{p\}$, then $s_p \psi$ is a central $0-1$ state on L and $s_p \psi(a) = 1$, $s_p \psi(b) = 0$.

Conversely, if L possesses weak hidden variables and if we denote by Ω the set of all central $0-1$ states, then a routine verification gives that it suffices to take for B the σ-algebra generated by all sets $\Omega_a = \{h, h(a) = 1\}$ $(a \in L)$ and put $\psi(a) = \Omega_a$. This completes the proof.

Now a natural question arises, whether each L possesses weak hidden variables (provided, of course, that $C(L)$ possesses weak hidden variables, which obviously requires $C(L)$ to have a set representation). The answer is in the negative.

Example 3. There exists a logic L such that

(i) $C(L)$ is σ-isomorphic to a σ-algebra of subsets of a set,

(ii) there exists $e \in L$ such that $s(e) = 1$ for no central $0-1$ state.

The construction. Let M be a six element logic $M = \{0, 1, a, a', b, b'\}$ and let S be a set with card $S = 2^N$. Put $L_x = M$ for any $x \in S$ and consider the logic product $P = \prod_{x \in S} L_x$ (the domain of P is the usual cartesian product and the partial ordering and the orthocomplement are taken "coordinatewise"). Let us define a relation $\sim$ on P by putting $f \sim g$ if and only if the following conditions are satisfied (elements of P are considered as mappings from S into L):

(i) $f^{-1}(b) = g^{-1}(b)$, $f^{-1}(b') = g^{-1}(b')$,

(ii) $f^{-1}(1) \cup f^{-1}(a') = g^{-1}(1) \cup g^{-1}(a')$,

(iii) $\{x \in S, f(x) \neq g(x)\}$ is at most countable.

Further, put $N_{f,g} = \{x \in S, f(x) \nleq g(x)\}$ and define another relation $\lesssim$ on P by setting $f \lesssim g \Leftrightarrow N_{f,g}$ is at most countable and $N_{f,g} \subset (f^{-1}(a) \cup g^{-1}(a'))$. The relation $\sim$ on P is an equivalence and the factor $P = L/\sim$ becomes a logic when endowed with the partial ordering and the orthocomplement induced by $\lesssim$ and ', respectively (the verification of these facts is rather lengthy but essentially simple and is left to the reader).

Now we have to show that $C(L)$ is isomorphic to a σ-algebra of subsets of a set. In order to do so, observe that $[f] \in C(L)$ $(f \in P)$ exactly in the case when the set $\{x \in S, f(x) \notin \{0, 1\}\}$ is countable. It immediately follows that the mappings s_x, r_x $(x \in S)$: $C(L) \to \{0, 1\}$ defined by the requirements

$$s_x([f]) = 1 \quad \text{if and only if} \quad f(x) \in \{1, a', b\},$$
$$r_x([f]) = 1 \quad \text{if and only if} \quad f(x) \in \{1, a', b'\}$$

are $0-1$ measures on $C(L)$. This implies that for any $[f] \in C(L)$ there is a $0-1$ measure t on $C(L)$ with $t([f]) = 1$. Therefore, $C(L)$ has a set representation.

Finally, put $e = [f_a]$, where $f_a(x) = a$ for any $x \in S$. We have to show that there is no central $0-1$ state h on L with $h(e) = 1$. Assume that such an h exists and proceed by way of contradiction. For each $K \subset S$, let f_K be the characteristic function of K (with 0, 1 taken from M). The mapping φ: $K \to [f_K]$ is an isomorphism of the Boolean algebra $\exp S$ (of all subsets of S) onto a sub-σ-algebra of $C(L)$. Therefore $m = h \circ \varphi$ is a probability measure on $\exp S$. Obviously, if $K \in \exp S$ and $S \setminus K$ is countable, then $[f_K] \geqq [f_a]$ and therefore $m(K) = h([f_K]) \geqq h([f_a]) = 1$. This implies that m is a two-valued probability measure on $\exp S$ such that $m(J) = 0$ for each countable set $J \in \exp S$. We have reached a contradiction (see [10]). The proof is complete.

In the conclusion of this note let us observe that the above example has the following central properties potentially applicable also elsewhere:

(i) We have $\bigwedge\{[f_K], K \subset \exp S, K \text{ countable}\} = 0$ in $C(L)$ but $0 = [f_a] \leqq f_K$ for any K countable.

(ii) $C(L)$ is atomic, the intersection of $[f_a]$ with every atom in $C(L)$ equals 0 but $[f_a] \neq 0$.

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Souhrn

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POZNÁMKA O SLABÝCH SKRYTÝCH PARAMETRECH

Článek se zabývá σ-aditivní verzí centrálně aditivních skrytých parametrů zavedených v [9]. Je nalezena logika, která nemá úplnou množinu centrálně aditivních bezdisperzních stavů.

Резюме

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ЗАМЕЧАНИЕ О СЛАБЫХ СКРЫТЫХ ПАРАМЕТРАХ

Рассматриваются центральные состояния на логике, введеные в связи с проблемой скрытых параметров. Построена логика, не имеющая полное семейство центральных 0—1 состояний.

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