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CONTINUOUS DEPENDENCE ON A PARAMETER OF SOLUTIONS OF GENERALIZED DIFFERENTIAL EQUATIONS

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Summary. In the theory of generalized differential equations an interesting convergence effect occurs which was described by J. Kurzweil as the R-emphatic convergence. Using the notion of a generalized differential equation with a substitution, so-called convergence under substitution will be defined and will appear to be very similar to the R-emphatic convergence. A sequence of equations which is convergent under substitution can be transformed to another sequence of equations which converges to its limit equation in a classical way, i.e. with the uniform convergence of solutions and of right-hand sides of these equations.

Keywords: generalized differential equation, generalized differential equation with a substitution, continuous dependence on a parameter, R-emphatic convergence, convergence under substitution.

AMS classification: 34A10, 34C20.

INTRODUCTION

If we study the behavior of solutions of a sequence of ordinary differential equations

\[ \frac{dx}{dt} = f(x, t) + g(x) \varphi_k(t) \]

where the functions \( \varphi_k \) “tend to the Dirac function”, we find that the classical continuous dependence theorems cannot be used. J. Kurzweil investigated this problem in 1958 in his paper [K2] and introduced the so-called R-emphatic convergence of the right-hand sides of generalized differential equations

\[ \frac{dx}{d\tau} = DF_k(x, t), \]

which ensures the pointwise convergence of solutions of these equations.

In this paper an auxiliary notion of the generalized differential equation with a substitution

\[ x(t) = y(\nu(t)), \quad \frac{dy}{d\tau'} = DH(y, \tau') \]
1. THE GENERALIZED DIFFERENTIAL EQUATION WITH A SUBSTITUTION

1.1. Let $\mathbb{N}$ denote the set of all positive integers, let $\mathbb{R}^N (N \in \mathbb{N})$ be the $N$-dimensional Euclidean space with the norm $|\cdot|$, $\mathbb{R}^1 = \mathbb{R}$. The symbol $(a_n)_{n=0}^{\infty}$ denotes a sequence.

1.2. If a function $g: [a, b] \rightarrow \mathbb{R}^N$, $-\infty < a < b < +\infty$, is of bounded variation, it can be written as a sum of its continuous and jump parts; these will be denoted by $g^C$, $g^J$, respectively. We assume that $g^C(a) = g(a)$, $g^J(a) = 0$.

We will write $g(t-) = \lim_{t \rightarrow t-} g(\tau)$, $g(t+) = \lim_{t \rightarrow t+} g(\tau)$, if the limits exist. The symbol $g(v(t+))$ denotes the same as $g(s+)$ where $s = v(t+)$. If $v$ is an increasing function, then evidently $g(v(t+)) = \lim_{t \rightarrow t+} g(v(\tau))$ provided the left-hand side has sense.

1.3. A function $x: [a, b] \rightarrow \mathbb{R}^N$ is called regulated if the onesided limits $x(t-)$ and $x(t+)$ exist and are finite for all $t \in (a, b]$ and $t \in [a, b)$, respectively. Since every regulated function $x: [a, b] \rightarrow \mathbb{R}^N$ is bounded, we may denote $\|x\| = \sup \{|x(t)|; t \in [a, b]\}$.

Let us denote by $\mathcal{R}_N[a, b]$ the normed linear space of all regulated functions from $[a, b]$ to $\mathbb{R}^N$ with the norm $\|\cdot\|$.

Then $\mathcal{R}_N[a, b]$ is a Banach space. For information about regulated functions see [F2].

1.4. A set $\mathcal{A} \subset \mathcal{R}_N[a, b]$ is called equiregulated if it has the following property:

For every $\varepsilon > 0$ and $t_0 \in [a, b]$ there is $\delta > 0$ such that

(i) if $x \in \mathcal{A}$, $t' \in [a, b]$ and $t_0 - \delta < t' < t_0$, then $|x(t_0-) - x(t')| < \varepsilon$,

(ii) if $x \in \mathcal{A}$, $t'' \in [a, b]$ and $t_0 < t'' < t_0 + \delta$, then $|x(t'') - x(t_0+)| < \varepsilon$.

In [F2], Th. 2.18 it is proved that for a set $\mathcal{A} \subset \mathcal{R}_N[a, b]$ the following conditions are equivalent:

(i) $\mathcal{A}$ is relatively compact in $\mathcal{R}_N[a, b]$;

(ii) $\mathcal{A}$ is equiregulated and for every $t \in [a, b]$ the set $\{x(t); x \in \mathcal{A}\}$ is bounded;

(iii) the set $\{x(a); x \in \mathcal{A}\}$ is bounded and there is an increasing continuous function $\eta: [0, \infty) \rightarrow [0, \infty)$, $\eta(0) = 0$ and an increasing function $K: [a, b] \rightarrow \mathbb{R}$ such that

$|x(t_2) - x(t_1)| \leq \eta(K(t_2) - K(t_1))$ for every $x \in \mathcal{A}$, $a \leq t_1 < t_2 \leq b$.

1.5. In this paper we will use the generalized Perron integral, which was introduced by J. Kurzweil in [K1]. A treatise of this integral which is sufficient for our purposes can be found in [S1]. We will use the notation from [S1].
A finite sequence of numbers \( A = \{ \alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \alpha_k \} \) is called a partition of the interval \([a, b]\) if
\[
a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b \quad \text{and} \quad \alpha_{i-1} \leq \tau_i \leq \alpha_i, \quad i = 1, 2, \ldots, k.
\]
Given a function \( \delta: [a, b] \rightarrow (0, \infty) \), we denote by \( \mathcal{A}(\delta) \) the set of all partitions \( A \) such that
\[
[\alpha_{i-1}, \alpha_i] \subset [\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)] \quad \text{for} \quad i = 1, 2, \ldots, k.
\]
The symbol \( \mathcal{S}[a, b] \) denotes the system of all sets \( S \subset [a, b] \times [a, b] \) satisfying the following condition: For every \( \tau \in [a, b] \) there is \( \delta(\tau) > 0 \) such that \( (\tau, t) \in S \) for every \( t \in [a, b] \cap [\tau - \delta(\tau), \tau + \delta(\tau)] \).

Let \( S \in \mathcal{S}[a, b] \), assume that a function \( U: S \rightarrow \mathbb{R} \) is given. If \( \delta \) is a function on \([a, b]\) which corresponds to \( S \) then for every partition \( A \in \mathcal{A}(\delta), A = \{ \alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \alpha_k \} \) the finite sum \( s(U, A) = \sum_{i=1}^{k}[U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})] \) is defined; \( s(U, A) \) is the integral sum corresponding to the function \( U \) and the partition \( A \).

A function \( U: S \rightarrow \mathbb{R}^{N}, S \in \mathcal{S}[a, b] \) is called integrable over \([a, b]\) if there exists \( \gamma \in \mathbb{R}^{N} \) such that for every \( \epsilon > 0 \) there exists \( \delta: [a, b] \rightarrow (0, \infty) \) such that for every \( A \in \mathcal{A}(\delta) \) the inequality \( |s(U, A) - \gamma| < \epsilon \) holds. The element \( \gamma \in \mathbb{R}^{N} \) is called the generalized Perron integral of \( U \) over the interval \([a, b]\) and will be denoted by \( s_{P}DU(\tau, t) \). If \( s_{P}DU(\tau, t) \) exists then we define \( s_{P}DU(\tau, t) = -s_{P}DU(t, \tau) \). We set \( s_{P}DU(\tau, t) = 0 \) if \( a = b \).

In [K1], Def. 1.1.1 and Def. 1.1.4 an equivalent definition of the generalized Perron integral is given (the equivalence is proved in Th. 1.2.1 in [K1]). This definition can be formulated as follows:

The function \( U: S \rightarrow \mathbb{R}, S \in \mathcal{S}[a, b] \) is integrable over \([a, b]\) and has the integral \( \gamma \in \mathbb{R} \) if for every \( \epsilon > 0 \) there is \( \delta: [a, b] \rightarrow (0, \infty) \) and functions \( m, M: [a, b] \rightarrow \mathbb{R} \) such that \( \gamma - \epsilon < m(b) - m(a) \leq M(b) - M(a) < \gamma + \epsilon \) and \((t - \tau)[m(t) - m(\tau)] \leq (t - \tau)[U(\tau, t) - U(\tau, t)] \leq (t - \tau)[M(t) - M(\tau)] \) for every \((\tau, t) \in S \) such that \( |\tau - t| < \delta(\tau) \).

This definition will be convenient for proving the following lemma:

1.6. Lemma. Let a function \( U: S \rightarrow \mathbb{R}, S \in \mathcal{S}[a, b] \) be given, assume that there is a nondecreasing function \( h^*: [a, b] \rightarrow \mathbb{R} \) which has zero continuous part, is left-continuous on \((a, b)\) and such that \( |U(\tau, t) - U(\tau, t)| \leq |h^*(t) - h^*(\tau)| \) for every \((\tau, t) \in S \). Then the function \( U \) is integrable over \([a, b]\) and
\[
\int_{a}^{b}DU(\tau, t) = \sum_{a \leq \tau < b} [U(t, t+) - U(t, t)].
\]

Proof. For every \( t \in [a, b] \) the limit \( U(t, t+) \) exists because \( |U(t, s^*) - U(t, s^*)| \leq \leq |h^*(s^*) - h^*(s^*)| \) if \((t, s^*), (t, s^*) \in S, t < s^* < s^*\). Denote \( \alpha_t = U(t, t+) - U(t, t) \).

Owing to the estimate \( |\alpha_t| \leq h^*(t+) - h^*(t) \) the series \( \sum_{a \leq t < b} \alpha_t \) is absolutely convergent; let its sum be denoted by \( \gamma \).

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Let $\epsilon > 0$ be given. There are points $a = t_1 < t_2 < \ldots < t_{k+1} = b$ such that
\[
\sum_{t \in (a, b) \setminus \{t_1, \ldots, t_k\}} [h^*(t - h^*(t))] = \sum_{i=1}^k [h^*(t_{i+1}) - h^*(t_i + 1)] < \frac{\epsilon}{2}.
\]
Define $\delta(\tau) = \min \{|\tau - t_i|, i = 1, 2, \ldots, k + 1\}$ for $\tau \in [a, b] \setminus \{t_1, t_2, \ldots, t_k\}$;
\[
\delta(t_j) = \min \{|t_i - t_j|; i = 1, 2, \ldots, k + 1, i \neq j\} \quad \text{for} \quad j = 1, 2, \ldots, k.
\]
If we define $\chi(t) = \sum_{s \in [a, t] \setminus \{t_1, \ldots, t_k\}} [h^*(s - h^*(s))]$, $t \in [a, b]$ then the function $\chi$ is nondecreasing and $\chi(b) - \chi(a) < \epsilon/2$. Let us define functions
\[
m(t) = \sum_{a \leq s < t} \alpha_s - 2 \chi(t), \quad M(t) = \sum_{a \leq s < t} \alpha_s + 2 \chi(t), \quad t \in [a, b].
\]
Then
\[
m(b) - m(a) = \gamma - 2[\chi(b) - \chi(a)] > \gamma - \epsilon; \\
M(b) - M(a) = \gamma + 2[\chi(b) - \chi(a)] < \gamma + \epsilon.
\]
If the pair $(\tau, t)$ belongs to $S$ and $\tau < t < \tau + \delta(\tau)$ then none of the points $t_1, t_2, \ldots, t_k$ belongs to the interval $(\tau, t)$. Hence $\chi(t) - \chi(\tau) = h^*(t) - h^*(\tau)$ provided $\tau \notin \{t_1, t_2, \ldots, t_k\}$ and $\chi(t) - \chi(\tau) = h^*(t) - h^*(\tau + 1)$ provided $\tau = t_j$ for some $j \in \{1, 2, \ldots, k\}$. We have the inequality
\[
U(\tau, t) - U(\tau, \tau) = [U(\tau, \tau + 1) - U(\tau, \tau)] + [U(\tau, t) - U(\tau, \tau + 1)] \leq \\
\leq \alpha_t + [h^* (t) - h^* (\tau + 1)] \leq \alpha_t + \sum_{t \leq s < t} \alpha_s + 2[h^* (t) - h^* (\tau + 1)] \leq \\
\leq \sum_{t \leq s < t} \alpha_s + 2[\chi(t) - \chi(\tau)] = M(t) - M(\tau).
\]
Similarly it can be proved that if $(\tau, t) \in S$ and $\tau - \delta(\tau) < t < \tau$ then
\[
U(\tau, t) - U(\tau, \tau) \leq h^*(t) - h^*(\tau) \leq M(\tau) - M(t).
\]
The inequality
\[
(t - \tau) [m(t) - m(\tau)] \leq (t - \tau) [U(\tau, t) - U(\tau, \tau)], \quad (\tau, t) \in S, \quad |t - \tau| < \delta(\tau)
\]
can be verified analogously.

1.7. In [S1], [S2] we can find basic results concerning the generalized differential equation
\[
(1.1) \quad \frac{dx}{dt} = DF(x, t).
\]
The function $F$ on the right-hand side of (1.1) is a vector-valued function from $G$ to $\mathbb{R}^N$, where $G$ is a subset of $\mathbb{R}^{N+1}$.

An $N$-vector valued function $x$ is a solution of the equation (1.1) on an interval $I \subset \mathbb{R}$, if $(x(t), t) \in G$ for all $t \in I$ and if for every $s_1, s_2 \in I$ the identity
holds. The integral used on the right-hand side of (1.2) is the generalized Perron integral of the function $U(x, t) = F(x(t), t)$.

Assume that $I, I'$ are intervals of the form $[t_0, t_0 + \sigma], [t_0, t_0 + \sigma']$ or $[t_0, t_0 + \sigma'), [t_0, t_0 + \sigma')$. Let $x, y$ be solutions of the equation (1.1) on the intervals $I, I'$, respectively. The solution $y$ is called a continuation of $x$ if $I \subset I'$ and if $x(t) = y(t)$ for every $t \in I$. If $I \neq I'$ then the solution $y$ is called a proper continuation of the solution $x$.

Solutions to which there is no proper continuation are called maximal solutions of (1.1).

1.8. Throughout this paper let $T > 0$ be a fixed number and $\Omega \subset \mathbb{R}^n$ a fixed open set. Denote $G = \Omega \times (-T, T)$.

Assume that $h, k, l: [-T, T] \rightarrow \mathbb{R}$ are nondecreasing functions which are continuous from the left on $(-T, T]$ and continuous from the right at the point $-T$, and let $\omega: [0, \infty) \rightarrow [0, \infty)$ be a continuous increasing function such that $\omega(0) = 0$.

We will be concerned with the class $\Phi(G, k, l, \omega)$ of functions $F$ occurring on the right-hand side of (1.1).

**Definition.** A function $F: G \rightarrow \mathbb{R}^n$ belongs to the class $\Phi(G, k, l, \omega)$ if

\[
(1.3) \quad |F(x, t_2) - F(x, t_1)| \leq |k(t_2) - k(t_1)| \quad \text{for all} \quad (x, t_1), (x, t_2) \in G,
\]

\[
(1.4) \quad |F(x, t_2) - F(x, t_1) - F(y, t_2) + F(y, t_1)| \leq \omega(|x - y|) |l(t_2) - l(t_1)|
\]

for all $(x, t_1), (x, t_2), (y, t_1), (y, t_2) \in G$.

We denote $\mathcal{F}(G, h, \omega) = \Phi(G, h, h, \omega)$.

Whenever the symbol $\Phi(G, k, l, \omega)$ or $\mathcal{F}(G, h, \omega)$ is used in this paper, it will be assumed that the set $G$ and the functions $h, k, l, \omega$ have the properties described above.

1.9. Remark. (i) In [K1], [K2] and [S1], [S2] the set $\mathcal{F}(G, h, \omega)$ is used except Chap. 5 in [S2]. For one function $F$ this is not important since if $F \in \Phi(G, k, l, \omega)$ and we denote $h(t) = k(t) + l(t)$ then $F \in \mathcal{F}(G, h, \omega)$. Nevertheless, to distinguish the two functions $k$ and $l$ is of importance when one is concerned with an infinite set of such functions $F$.

(ii) The continuity at the endpoints of the interval $[-T, T]$ of functions $h, k, l$ is assumed only for technical purposes. Since we will work on the set $G = \Omega \times (-T, T)$, nothing changes if e.g. a function $h$ is only left-continuous on $(-T, T)$; it can be re-defined by the value $h((-T)^+)$ at the point $-T$ and by $h(T^-)$ at $T$. 

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1.10. **Remark.** It follows from (1.3) that for every \( x \in \Omega \) the function \( F(x, \cdot) \) has bounded variation on \((-T, T)\). If a function \( x \) is a solution of (1.1) on \([a, b]\) then
\[
|x(t_2) - x(t_1)| \leq k(t_2) - k(t_1), \quad a \leq t_1 < t_2 \leq b
\]
according to Lemma 2.6 in \([S1]\); hence the function \( x \) has bounded variation.

1.11. **Let us denote by** \( V^- \) **the set of all increasing functions** \( v: [-T, T] \to [-T, T] \)
**which are continuous from the left on** \((-T, T]\) **and continuous from the right at**
**the point** \(-T, v(-T) = -T, v(T) = T\).

**By** \( \Lambda \) **let us denote the set of all functions** \( \lambda: [-T, T] \to [-T, T] \)
**which are continuous and increasing on** \([-T, T]\), \( \Lambda(-T) = -T, \Lambda(T) = T \).

1.12. **Definition.** Assume that functions \( H \in \Phi(G, k, l, \omega) \) and \( v \in V^- \) are given,
**let** \( I \subset (-T, T) \) **be an interval.**

(i) **An** \( N \)-vector valued function \( x \) **is a solution of the generalized differential equation**
**with a substitution**
\[
(1.5) \quad x(t) = y(v(t)), \quad \frac{dy}{dt'} = DH(y, t')
\]
on the interval \( I \), if there exists an interval \( J \subset (-T, T) \) and a solution \( y \) of the
generalized differential equation
\[
(1.6) \quad \frac{dy}{dt'} = DH(y, t')
\]
on the interval \( J \) such that the equality \( x(t) = y(v(t)) \) holds for every \( t \in I \).

(ii) **We say that the solution** \( x \) **is a maximal solution of** (1.5) **if it has no proper
continuation** (defined as in 1.7).

(iii) **Let** \( x \) **be a solution of** (1.5) **on** \([t_0, c]\) **and let** \( v(c) < v(c+) \). **We say that** \( x \)
disappears at the point \( c \) if \( x(t) = y(v(t)) \) holds on \([t_0, c]\) for some maximal solution \( y \)
of (1.6) which is defined on an interval \( J \) such that its right endpoint belongs to
\([v(c), v(c+)]\) and \( v(c) \in J \).

1.13. **Remark.** It is possible that a solution \( x \) disappears at a point \( c \) but it can be
continued to the right. This situation occurs when there are two solutions \( y_1, y_2 \)
of (1.6) on intervals \( J_1, J_2 \), respectively, such that \( J_1 \supseteq [v(t_0), v(c+)], J_2 = [v(t_0), d] \)
or \([v(t_0), d] \) where \( d \in \([v(c), v(c+)]\), \( y_2 \) is a maximal solution of (1.6) on \( J_2 \) and
\( x(t) = y_1(v(t)) = y_2(v(t)) \) for every \( t \in [t_0, c] \).

1.14. **Example.** Assume that \( T = 2, H(y, t) = yt; v(t) = t \) for \( t \in [-2, 0] \) and
\( v(t) = 1 + t/2 \) for \( t \in (0, 2] \). By \([S1]\), Chap. 4A the equation with substitution (1.5)
can be written in the form
\[
(1.7) \quad x(t) = y(v(t)), \quad \frac{dy}{ds} = y
\]
and its solutions are the functions \( x(t) = x_0e^{t}, t \in [-2, 0], x(t) = x_0e^{1+t/2}, t \in (0, 2] \).
1.15. Definition. Assume that \( F \in \Phi(G, k, l, \omega) \), \( v \in A \) are given. A function \( H: G \to \mathbb{R}^N \) is called the prolongation of the function \( F \) along \( v \), if

\[
H(x, t) = F(x, t) \quad \text{for every} \quad (x, t) \in G.
\]

1.16. Proposition. Assume that \( F \in \Phi(G, k, l, \omega) \) and \( v \in A \) are given, let the function \( H: G \to \mathbb{R}^N \) be the prolongation of \( F \) along \( v \). Then \( H \in \Phi(G, k \circ v^{-1}, l \circ v^{-1}, \omega) \).

The proof is evident.

1.17. Theorem. Assume that functions \( F \in \Phi(G, k, l, \omega) \) and \( v \in A \) are given, let \( H \in \Phi(G, k', l', \omega) \) be the prolongation of \( F \) along the function \( v \). Then the equations (1.1) and (1.5) have the same solutions.

Proof. First assume that \( x \) is a solution of (1.1) on \( I \) and define \( J = \{v(t); v \in I\} \), \( y(t') = x(v^{-1}(t')) \) for every \( t' \in J \). For every \( \sigma_1, \sigma_2 \in J \) we have

\[
y(\sigma_2) - y(\sigma_1) = x(v^{-1}(\sigma_2)) - x(v^{-1}(\sigma_1)) = \int_{v^{-1}(\sigma_1)}^{v^{-1}(\sigma_2)} \, DF(x(\tau), t) \, d\tau.
\]

By Th. 1.24 in [S1] we conclude that

\[
\int_{v^{-1}(\sigma_1)}^{v^{-1}(\sigma_2)} \, DF(x(\tau), t) = \int_{\sigma_1}^{\sigma_2} \, DF(x(v^{-1}(\tau)), v^{-1}(t)) = \int_{\sigma_1}^{\sigma_2} \, DH(y(\tau), t) \, d\tau.
\]

This means that the function \( y \) is a solution of (1.6) on \( J \) and consequently the function \( x \) is a solution of (1.5) on \( I \).

On the other hand, if the function \( x \) is a solution of (1.5) on \( I \) then there is a solution \( y \) of the equation (1.6) on \( J \) such that \( x(t) = y(v(t)) \) for \( t \in I \). Th. 1.24 in [S1] implies that

\[
x(t_2) - x(t_1) = y(v(t_2)) - y(v(t_1)) = \int_{v(t_1)}^{v(t_2)} \, DH(y(\tau), t) = \int_{t_1}^{t_2} \, DF(x(\tau), t) \quad \text{for every} \quad t_1, t_2 \in I.
\]

1.18. Let functions \( H \in \Phi(G, k, l, \omega) \) and \( v \in V^* \) be given. By \( R_{(H,v)} \) we denote the set of all pairs \((x, t)\in G\) with the following properties:

(i) If \( v \) is continuous at \( t \) then \( x + H(x, v(t) +) - H(x, v(t)) \in \Omega \); let us denote \( p(x, t) = 0 \).

(ii) if \( v(t) < v(t+) \) then there exist \( \delta > 0 \) and a unique solution \( y \) of the initial value problem

\[
\frac{dy}{dt'} = DH(y, t'), \quad y(v(t)) = x
\]
on the interval \([v(t), v(t + \delta)]\). Moreover, 

\[
(1.9) \ \text{there exists } \delta > 0 \ \text{such that } \ z \in \Omega \ \text{for any } \ z \in \mathbb{R}^N \ \text{satisfying } \ |z - y(s)| < \delta \ \text{for some } \ s \in [v(t), v(t + \delta)].
\]

Denote \(p(x, t) = y(v(t+) +) - x\).

1.19. Proposition. Assume that functions \(H \in \Phi(G, k, l, \omega)\) and \(v \in V^-\) are given. Then

(i) \(|p(x, t)| \leq k(v(t+) +) - k(v(t))\) for every \((x, t) \in R_{(H, v)}\);

(ii) for every \((x, t) \in G\) the series

\[
\sum_{-T < s < t} \left[ p(x, s) - H(x, v(s+) +) + H(x, v(s)) \right]
\]

is absolutely convergent.

Proof. (i) Using Lemma 2.5 in [S1] we get the estimate

\[
|p(x, t)| = |y(v(t+) +) - x| = \lim_{s \to t^+} |y(v(s)) - x| = \lim_{s \to t^+} \left| \int_{v(s)}^{v(t)} DH(y(\tau^'), t') \right| \leq
\]

\[
\leq \lim_{s \to t^+} \left[ k(v(s)) - k(v(t)) \right] = k(v(t+) +) - k(v(t)).
\]

(ii) Since the composition \(k \circ v\) is a nondecreasing function, the set of all its points of discontinuity is at most countable. Hence there is a sequence \((s_j)_{j=1}^\infty\) of pairwise different points from \((-T, T)\) such that \(k(v(t+) +) = k(v(t))\) for every \(t \in (-T, T) \setminus \{s_1, s_2, \ldots\}\). For every \((x, t) \in R_{(H, v)}\) we have

\[
|p(x, t) - H(x, v(t+) +) + H(x, v(t))| \leq
\]

\[
\leq |p(x, t)| + |H(x, v(t+) +) - H(x, v(t))| \leq 2[k(v(t+) +) - k(v(t))];
\]

hence

\[
\sum_{(x, s) \in R_{(H, v)}} |p(x, s) - H(x, v(s+) +) + H(x, v(s))| \leq
\]

\[
\leq 2 \sum_{-T < s < T} [k(v(s+) +) - k(v(s))] = 2 \sum_{j=1}^\infty [k(v(s_j) +) - k(v(s_j))] \leq
\]

\[
\leq 2[k(v(T)) - k(v(-T))].
\]

1.20. Definition. Assume that functions \(H \in \Phi(G, k, l, \omega)\), \(v \in V^-\) are given. The function

\[
(1.10) \quad F(x, t) = H(x, v(t)) + \\
+ \sum_{-T < s < t} \left[ p(x, s) - H(x, v(s+) +) + H(x, v(s)) \right], \quad (x, t) \in G
\]

is called the reduction of the function \(H\) by the function \(v\).
1.21. Proposition. Assume that the function $F: G \to \mathbb{R}^N$ is the reduction of a function $H \in \mathcal{F}(G, k, l, \omega)$ by a function $v \in V^-$. Define

$$h(t) = 2 \sum_{\frac{T}{T} \leq s < t \leq \frac{T}{T}} [k(v(s+) +) - k(v(s))], \; t \in [-T, T].$$

Then

$$(1.11) \quad |F(x, t_2) - F(x, t_1) - H(x, v(t_2)) + H(x, v(t_1))| \leq |h(t_2) - h(t_1)|$$

for every $(x, t_1), (x, t_2) \in G$.

Proof. The proposition follows immediately from the proof of Prop. 1.19.

1.22. Example. Let us return to Example 1.14. In this case the reduction of the function $H$ by the function $v$ will have the form $F(x, t) = xt$ for $t \in [-2, 0]$, $F(x, t) = x(e - 1 + t/2)$ for $t \in (0, 2]$.

1.23. Lemma. Let functions $H \in \mathcal{F}(G, k, l, \omega)$ and $v \in V^-$ be given, assume that $F$ is the reduction of the function $H$ by $v$. Assume that the function $y: [v(a), v(b)] \to \mathbb{R}^N$ ($-T < a < b < T$) satisfies the following conditions:

(i) $(y(v(t)), t) \in R_{(H, \omega)}$ for every $t \in [a, b]$;
(ii) the function $y \circ v$ is regulated;
(iii) the integral $\int_{v(a)}^{v(b)} DH(y(\tau'), \tau')$ exists for every $s \in [a, b]$.

Then the integrals $\int_{v(a)}^{v(b)} DF(y(v(\tau)), \tau)$ and $\int_{v(a)}^{v(b)} DH(y(\tau'), \tau')$ exist and the equality

$$(1.12) \quad \int_{v(a)}^{v(b)} DF(y(v(\tau)), \tau) - \int_{v(a)}^{v(b)} DH(y(\tau'), \tau') = \sum_{a \leq s < b} \left[ F(y(v(s)), s +) - F(y(v(s)), s) - \int_{v(s)}^{v(s+)} DH(y(\tau'), \tau') - H(y(v(s+)), v(s+)) + H(y(v(\tau)), v(\tau)) \right]$$

holds.

Proof. Since the function $H(x, v(t))$ obviously belongs to $\mathcal{F}(G, h, \omega)$ with $h(t) = k(v(t)) + l(v(t))$, the existence of the integral $\int_{a}^{b} DH(y(v(\tau)), v(\tau))$ follows from Corollary 2.11 in [S1].

All assumptions of the Theorem in [F1] being satisfied, the existence of $\int_{a}^{b} DH(y(v(\tau)), v(\tau))$ implies that the integral $\int_{v(a)}^{v(b)} DH(y(\tau'), \tau')$ exists and the equality

$$(1.13) \quad \int_{a}^{b} DH(y(v(\tau)), v(\tau)) - \int_{v(a)}^{v(b)} DH(y(\tau'), \tau') = \sum_{a \leq s < b} \left[ H(y(v(s)), v(s+)) + H(y(v(\tau)), v(\tau)) - H(y(v(s+)), v(s+)) + H(y(v(\tau)), v(\tau)) \right]$$

holds.
holds. Let us denote \( F^*(x, t) = F(x, t) - H(x, v(t)) \) for \((x, t) \in G\). By Proposition 1.21 the assumptions of Lemma 1.6 are fulfilled for every component \( [F^*(y(v(\tau)), t)] = U(\tau, t), j = 1, 2, \ldots, N\) of the vector-valued function \( F^*(y(v(\tau)), t)\). Consequently, the integral \( \int_a^b DF^*(y(v(\tau)), t) \) exists and we have

\[
\int_a^b DF^*(y(v(\tau)), t) - H(y(v(\tau)), v(t))] = \int_a^b DF^*(y(v(\tau)), t) = \sum_{s \leq t < b} [F^*(y(v(s)), s +) - F^*(y(v(s)), s)] + F(y(v(s)), s) - H(y(v(s)), v(s +)) = H(y(v(s)), v(s +)) + H(y(v(s)), v(s)) - F(y(v(s)), s).
\]

From the existence of the integrals \( \int_a^b DF^*(y(v(\tau)), t) \) and \( \int_a^b DH(y(v(t)), v(t)) \) we conclude that the integral \( \int_a^b DF(y(v(t)), t) \) exists. Combining (1.13) and (1.14) we get the equality (1.12).

1.24. Theorem. Let functions \( H \in \Phi(G, k, l, \omega) \) and \( v \in V^- \) be given, assume that the function \( F: G \to \mathbb{R}^N \) is the reduction of \( H \) by \( v \). Assume that a function \( x: [a, b] \to \mathbb{R}^N \) is given such that \((x(t), t) \in R_{(H,v)} \) for every \( t \in [a, b] \).

Then the function \( x \) is a solution of the equation (1.1) on \([a, b]\) if and only if it is a solution of the equation (1.5) on \([a, b]\).

Proof. Let \( x \) be a solution of (1.1) on \([a, b]\). By Lemma 2.6 in [S1] the function \( x \) is of bounded variation. Let us define a function \( y: [v(a), v(b)] \to \mathbb{R}^N \) in the following way:

For every \( \sigma \) such that \( \sigma = v(t) \) for some \( t \in [a, b] \) let us define \( y(\sigma) = x(t) \).

If \( t \in [a, b] \) is such that \( v(t) < v(t^+) \), then \((x(t), t) \in R_{(H,v)} \) by the assumption of this proposition, and therefore by 1.18 (ii) there exist \( \delta_t > 0 \) and an \( N\)-vector valued function \( y_t \), which is a solution of the initial value problem

\[
\frac{dy}{dt} = DH(y, t'), \quad y(v(t)) = x(t)
\]
on the interval \([v(t), v(t + \delta_t)]\). By 1.18 (ii) we have \( p(x, t) = y_t(v(t^+) +) - x \). It follows from (1.10) that \( F(x(t), t^+) - F(x(t), t) = p(x(t), t) \). Consequently,

\[
F(x(t), t^+) - F(x(t), t) = y_t(v(t^+) +) - x(t) = \lim_{s \to t^+} \int_{v(t)}^{v(s)} DH(y_t(t'), t') = \int_{v(t)}^{v(t^+)} DH(y_t(t'), t') + H(y_t(v(t^+) +), v(t^+) +) - H(y_t(v(t^+) +), v(t^+))
\]

(here Th. 1.15 from [S1] was used).

Now let us define \( y(\sigma) = y_t(\sigma) \) for every \( \sigma \in [v(t), v(t^+)] \).

Lemma 1.23 implies that for every \( s_1, s_2 \in [a, b] \) the integral \( \int_{v(s_1)}^{v(s_2)} DH(y(t'), t') \) exists. By (1.15) the sum on the right-hand side of the relation (1.12) is zero if \( a, b \) are replaced by \( s_1, s_2 \). Hence

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Assume that \( v(\alpha) \leq \sigma_1 < \sigma_2 \leq v(\beta) \). Let us find \( s_1, s_2 \in [\alpha, \beta] \) such that \( v(s_i) \leq \sigma_i \leq v(s_i^+) \), \( i = 1, 2 \). If \( s_1 = s_2 \) then we have the equality

\[
y(\sigma_2) - y(\sigma_1) = y_{s_1}(\sigma_2) - y_{s_1}(\sigma_1) = \int_{\sigma_1}^{\sigma_2} DH(y(\tau'), \tau') = \int_{\sigma_1}^{\sigma_2} DH(y(\tau'), \tau').
\]

If \( s_1 < s_2 \) then

\[
y(\sigma_2) - y(\sigma_1) = [y(\sigma_2) - y(v(s_2))] + [y(v(s_2)) - y(v(s_1))] + [y(v(s_1)) - y(\sigma_1)] + [y_{s_1}(\sigma_2) - y_{s_1}(v(s_2))] + [y_{s_1}(v(s_1)) - y_{s_1}(\sigma_1)] = \\
= \int_{\sigma_1}^{\sigma_2} DH(y_{s_1}(\tau'), \tau') + \int_{v(s_1)}^{v(s_2)} DH(y(\tau'), \tau') - \int_{v(s_1)}^{v(s_2)} DH(y_{s_1}(\tau'), \tau') = \\
= \int_{\sigma_1}^{\sigma_2} DH(y(\tau'), \tau').
\]

From (1.16), (1.17) and (1.18) it follows that the function \( y \) is a solution of (1.6) on the interval \([v(\alpha), v(\beta)]\), and consequently the function \( x \) is a solution of (1.5) on \([\alpha, \beta]\).

(ii) If the function \( x \) is a solution of the equation (1.5) then by Definition 1.12 there is a solution \( y \) of the equation (1.6) on \([v(\alpha), v(\beta)]\) such that \( x(t) = y(v(t)) \) for every \( t \in [\alpha, \beta] \). Analogously as in part (i) we conclude from Lemma 1.23 that

\[
x(s_2) - x(s_1) = y(v(s_2)) - y(v(s_1)) = \int_{v(s_1)}^{v(s_2)} DH(y(\tau'), \tau') = \int_{\sigma_1}^{\sigma_2} DF(x(\tau), \tau)
\]

for every \( s_1, s_2 \in [\alpha, \beta] \), which implies that the function \( x \) is a solution of (1.1) on \([\alpha, \beta]\).

2. CLASSICAL-CONTINUOUS DEPENDENCE THEOREMS

2.1. Lemma. Assume that a function \( F \in \Phi(G, k, l, \omega) \) is given. Then for every two regulated functions \( x, y: [\alpha, \beta] \to \Omega (-T < \alpha < \beta < T) \) the inequality

\[
|\int_{\alpha}^{\beta} D[F(x(\tau), \tau) - F(y(\tau), \tau)]| \leq \omega(\|x - y\|)(l(\beta) - l(\alpha))
\]

holds.
Proof. The integral in (2.1) exists owing to Corollary 2.11 in [S1]. By (1.4) we have
\[ |t - \tau| F(x(\tau), t) - F(y(\tau), t) - F(x(\tau), \tau) + F(y(\tau), \tau)| \leq (t - \tau) \omega(|x(\tau) - y(\tau)|) (l(t) - l(\tau)) \leq (t - \tau) \omega(\|x - y\|) (l(t) - l(\tau)) \]
for every \( \tau, \ t \in [\alpha, \beta] \). Corollary 1.18 in [S1] implies that
\[
\left| \int_{a}^{b} D[F(x(\tau), t) - F(y(\tau), t)] \right| \leq \int_{a}^{b} \omega(\|x - y\|) (l(\beta) - l(\alpha)).
\]

2.2. Lemma. Let a sequence of functions \( F_n \in \Phi(G, k_n, l_n, \omega) \), \( n = 0, 1, 2, \ldots \) be given; assume that
\[
\text{(2.2)} \quad \exists c > 0 \text{ such that } l_n(T) - l_n(-T) \leq c \quad \text{for every } n = 0, 1, 2, \ldots ;
\]
\[
\text{(2.3)} \quad F_n(x, t) \rightarrow F_0(x, t) \quad \text{and} \quad F_n(x, t+) \rightarrow F_0(x, t+) \quad \text{for every } (x, t) \in G.
\]
If \([a, b] \subset (-T, T)\) and if a function \( \varphi : [a, b] \rightarrow \Omega \) is constant on the open interval \((a, b)\), then
\[
\text{(2.4)} \quad \lim_{n \to \infty} \int_{a}^{b} DF_n(\varphi(t), t) = \int_{a}^{b} DF_0(\varphi(t), t).
\]

Proof. Assume that \( \varphi \) has a value \( d \) on \((a, b)\). From Th. 1.15 in [S1] we conclude that
\[
\text{(2.5)} \quad \int_{a}^{b} DF_n(\varphi(t), t) = F_n(d, b) - F_n(d, a+) + F_n(\varphi(a), a+) - F_n(\varphi(a), a)
\]
for \( n = 0, 1, 2, \ldots \). From (2.3) we then obtain (2.4).

2.3. Lemma. Assume that functions \( F_n \in \Phi(G, k_n, l_n, \omega) \), \( n = 0, 1, 2, \ldots \) satisfy
\[(2.2), (2.3)\). Then (2.4) holds for every finite step function \( \varphi : [a, b] \rightarrow \Omega \) \((-T < a < b < T)\).

Proof. Assume that \( \varphi \) has the form \( \varphi(\tau) = d_i \) for \( \tau \in (t_{i-1}, t_i) \), where \( a = t_0 < t_1 < \ldots < t_m = b \). Since
\[
\int_{a}^{b} DF_n(\varphi(\tau), t) = \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} DF_n(\varphi(\tau), t)
\]
for every \( n = 0, 1, 2, \ldots \), the relation (2.4) follows from Lemma 2.3.

2.4. Theorem. Assume that a sequence of functions \( F_n \in \Phi(G, k_n, l_n, \omega) \), \( n = 0, 1, 2, \ldots \) satisfies (2.2), (2.3).

Let \([\alpha, \beta] \subset (-T, T)\). For any \( n \in N \), let \( x_n \) be a solution to the equation
\[
\text{(2.6)} \quad \frac{dx}{d\tau} = DF_n(x, t)
\]
on \([a, \beta]\). Furthermore, let us assume that \(x_n\) tend uniformly on \([a, \beta]\) to such a function \(x_0\) that \(x_0(t) \in \Omega\) for any \(t \in [a, \beta]\).

Then the function \(x_0\) is a solution of the equation

\[
\frac{dx}{dt} = DF_0(x, t)
\]

on the interval \([a, \beta]\).

Proof. Since the functions \(x_n\) have bounded variations by [S1], Corollary 2.7, the function \(x_0\) is regulated on \([a, \beta]\); hence the integral \(\int_{t_1}^{t_2} DF_0(x_0(\tau), t)\) exists for every \(t_1, t_2 \in [a, \beta]\). By definition of solutions of (2.6) we have

\[
x_n(t_2) - x_n(t_1) = \int_{t_1}^{t_2} DF_n(x_n(\tau), t)
\]

for every \(n \in \mathbb{N}\).

If we prove that

\[
\lim_{n \to \infty} \int_{t_1}^{t_2} DF_n(x_n(\tau), t) = \int_{t_1}^{t_2} DF_0(x_0(\tau), t)
\]

we obtain the equality

\[
x_0(t_2) - x_0(t_1) = \int_{t_1}^{t_2} DF_0(x_0(\tau), t)
\]

for every \(t_1, t_2 \in [a, \beta]\), which implies that \(x_0\) is a solution of (2.7).

Let \(\epsilon > 0\) be given; let us find \(\lambda > 0\) such that \(\omega(\lambda) < \epsilon\). Since the function \(x_0\) is regulated, there is a finite step function \(\varphi: [a, \beta] \to \Omega\) such that \(\|x_0 - \varphi\| < \lambda/2\). Let \(n_1\) be such an integer that \(\|x_n - x_0\| < \lambda/2\) for every \(n \geq n_1\). By Lemma 2.3 there is \(n_2 \in \mathbb{N}\) such that

\[
\left| \int_{t_1}^{t_2} DF_n(\varphi(\tau), t) - \int_{t_1}^{t_2} DF_0(\varphi(\tau), t) \right| < \epsilon
\]

for every \(n \geq n_2\). Denote \(n_0 = \max(n_1, n_2)\).

For arbitrary \(n \geq n_0\) the inequality \(\|x_n - \varphi\| < \lambda\) holds; using Lemma 2.1 we get

\[
\left| \int_{t_1}^{t_2} DF_n(x_n(\tau), t) - \int_{t_1}^{t_2} DF_0(x_0(\tau), t) \right| \leq \int_{t_1}^{t_2} D[F_n(x_n(\tau), t) - F_0(\varphi(\tau), t)] + \int_{t_1}^{t_2} D[F_0(\varphi(\tau), t) - F_0(x_0(\tau), t)] \leq \\
\leq \omega(\|x_n - \varphi\|) (l_n(t_2) - l_n(t_1)) + \epsilon + \omega(\|\varphi - x_0\|) (l_0(t_2) - l_0(t_1)) \leq \\
\leq 2\omega(\eta) \epsilon + \epsilon < \epsilon(2c + 1).
\]

2.5. Theorem. Assume that a sequence of functions \(F_n \in \Phi(G, k_n, l_n, \omega)\), \(n = 0, 1, 2, \ldots\) satisfies (2.2);
There is a continuous increasing function \( \eta: [0, \infty) \rightarrow [0, \infty) \), \( \eta(0) = 0 \) and an increasing function \( K: [-T, T] \rightarrow \mathbb{R} \) which is left-continuous on \((-T, T]\), \( K(-T) = K((T) +) \) and such that
\[
k_n(t_2) - k_n(t_1) \leq \eta(K(t_2) - K(t_1)) \quad \text{for every } n \in \mathbb{N}, \quad -T \leq t_1 < t_2 \leq T;
\]
(2.9)
\[
F_n(x, t) \rightarrow F(x, t) \quad \text{for every } (x, t) \in G.
\]

(i) If for any \( n \in \mathbb{N} \) \( x_n \) is a solution of (2.6) on \( [\alpha, \beta] \) and the set \( \{x_n(\alpha), n \in \mathbb{N}\} \) is bounded, then the sequence \( (x_n)_{n=1}^\infty \) contains a subsequence which is convergent uniformly on \( [\alpha, \beta] \) to a function \( x_0 \in \mathcal{E}[\alpha, \beta] \).

(ii) If \( x_0(t) \in \Omega \) for every \( t \in [\alpha, \beta] \), then the function \( x_0 \) is a solution of (2.7) on \( [\alpha, \beta] \).

Proof. (i) By [S1], Lemma 2.6 we get from the assumption (2.8) that \( |x_n(t_2) - x_n(t_1)| \leq k_n(t_2) - k_n(t_1) \leq \eta(K(t_2) - K(t_1)) \) for every \( n \in \mathbb{N}, \alpha \leq t_1 < t_2 \leq \beta \).

According to Theorem 1.4 above equiregulated sets the functions \( x_1, x_2, \ldots \) are contained in a compact subset of \( \mathcal{E} \); hence there is a subsequence \( (x_{n_k})_{k=1}^\infty \)

which converges uniformly on \( [\alpha, \beta] \) to a function \( x_0 \in \mathcal{E}[\alpha, \beta] \).

(ii) Since \( |F_n(x, t_2) - F_n(x, t_1)| \leq k_n(t_2) - k_n(t_1) \leq \eta(K(t_2) - K(t_1)) \), we get by 1.4 that for every \( x \in \Omega \) the functions \( F_n(x, \cdot) \) uniformly converge to \( F_0(x, \cdot) \). This implies that (2.3) holds; now Theorem 2.4 can be used.

2.6. Theorem. Assume that functions \( F_n \in \Phi(G, k_n, l_n, \omega), n = 0, 1, 2, \ldots \) satisfy (2.2), (2.8) and (2.9).

Let an \( N \)-vector valued function \( x_0 \) be a solution of (2.7) on \([\alpha, \beta] \subset (-T, T)\) which has the following uniqueness property:

(2.10) If \( x \) is a solution of (2.7) on \( [\alpha, \gamma] \subset [\alpha, \beta] \) such that \( x(\alpha) = x_0(\alpha) \), then \( x(t) = x_0(t) \) for every \( t \in [\alpha, \gamma] \).

Assume further that

(2.11) there is \( q > 0 \) such that if \( y \in \mathbb{R}^N \), \( s \in [\alpha, \beta] \) and \( |y - x_0(s)| < q \) then \( y \in \Omega \).

Assume that a sequence \((y_n)_{n=1}^\infty \subset \mathbb{R}^N \) is given such that \( \lim_{n \to \infty} y_n = x_0(\alpha) \).

Then there is an integer \( n_0 \) such that for every \( n \geq n_0 \) there exists a solution \( x_n \)

of (2.6) on \( [\alpha, \beta] \), \( x_n(\alpha) = y_n \), and \( \lim x_n(t) = x_0(t) \) uniformly on \( [\alpha, \beta] \).

The proof is in fact the same as the proof of Theorem 2.4 in [S2], but under our assumptions which are somewhat more general it should rely on Theorem 2.5.

2.7. Corollary. Assume that functions \( F_n \in \Phi(G, k_n, l_n, \omega), n = 0, 1, 2, \ldots \) satisfy (2.2), (2.8) and (2.9). Let \( x_0 \) be a solution of the equation (2.7) on \([\alpha, \beta] \subset (-T, T)\) such that (2.10), (2.11) hold. Then for every \( \epsilon > 0 \) there is \( n_0 \in \mathbb{N} \) and \( \sigma > 0 \) such
that it holds: If \(x\) is a solution of the equation (2.6) on \([\alpha, \beta]\) for some \(n \geq n_0\) and if \(|x(\alpha) - x_0(\alpha)| < \sigma\), then \(\|x - x_0\| < \varepsilon\).

Proof. Assume that there is such \(\varepsilon_0 > 0\) that for every \(k \in \mathbb{N}\) there is \(n_k \geq k\) and such a solution \(x_k\) of (2.6) on \([\alpha, \beta]\) that \(|x_k(\alpha) - x_0(\alpha)| < 1/k\) and \(\|x_k - x_0\| \geq \varepsilon_0\). Then \(x_k(\alpha) \to x_0(\alpha)\); by Theorem 2.6 the sequence \((x_k)\) converges to \(x_0\) uniformly on \([\alpha, \beta]\), which is a contradiction.

3. THE R-EMPHATIC CONVERGENCE AND THE CONVERGENCE UNDER SUBSTITUTION

3.1. The concept of R-emphatic convergence of right-hand sides of generalized differential equations

\[ (3.1)_n \quad \frac{dx}{dt} = DF_n(x, t) \]

was introduced by J. Kurzweil in [K2].

Let a set \(R \subseteq G\) be given. Assume that for every \(n = 0, 1, 2, \ldots\) a function \(F_n \in \mathcal{F}(G, h_n, \omega)\) is given. The sequence \((F_n)_{n=1}^{\infty}\) converges R-emphatically to the function \(F_0\), if the following conditions are fulfilled:

\[ (3.2) \limsup_{n \to \infty} [h_n(t_2) - h_n(t_1)] \leq h_0(t_2) - h_0(t_1) \text{ if the function } h_0 \text{ is continuous at } t_1 \text{ and } t_2, -T < t_1 < t_2 < T; \]

\[ (3.3) \text{ there is a function } F^* : G \to \mathbb{R}^N \text{ such that } \]

\[ |F^*(x, t_2) - F^*(x, t_1)| \leq |h^*(t_2) - h^*(t_1)| \quad \text{for} \quad (x, t_1, (x, t_2) \in G \]

where \(h^*\) is the jump part of the function \(h_0\) and \(\lim_{n \to \infty} F_n(x, t) = F_0(x, t) + \]

\[ + F^*(x, t) \text{ if } (x, t) \in G \text{ and } t \text{ is a point of continuity of } h_0; \]

\[ (3.4) \text{ for every } (x_0, t_0) \in R \text{ the element } x_0 + F_0(x_0, t_0+) - F_0(x_0, t_0) \text{ belongs to } \Omega; \]

if, moreover, \(h_0(t_0+) > h_0(t_0)\), then for every \(\varepsilon > 0\) there is \(\delta > 0\) such that for each \(\delta' \in (0, \delta)\) there is \(n_0 \in \mathbb{N}\) with the following property: if \(x\) is a solution of (3.1) on \([t_0 - \delta', t_0 + \delta']\) for some \(n \geq n_0\) and if \(|x(t_0) - x_0| \leq \delta, \)

then \(|x(t_0 + \delta') - x(t_0 - \delta') - [F_0(x_0, t_0+) - F_0(x_0, t_0)]| < \varepsilon\).

The definition of R-emphatic convergence was invented so as to cover the problem of pointwise convergence of solutions of (3.1) to a solution of a limit equation

\[ (3.5) \quad \frac{dx}{dt} = DF_0(x, t). \]

In this chapter another type of convergence will be defined which will cover a similar convergence effect.
3.2. Definition. Assume that functions $F_n \in \Phi(G, k_n, l_n, \omega)$ are given for every $n \in \mathbb{N}$. Let functions $H: G \to \mathbb{R}^N$ and $v \in \mathcal{V}^-$ be given such that $H(x, \cdot)$ is left-continuous on $(-T, T]$ and right-continuous at $-T$.

We say that the functions $F_n$ converge under substitution to the pair $(H, v)$ if there exists a sequence of continuous increasing functions $v_n \in \Lambda$, $n \in \mathbb{N}$ such that the following conditions hold:

(i) $v_n(t) \to v(t)$ for every $t \in (-T, T)$ such that $v(t) = v(t^+)$;
(ii) there is $c > 0$ such that $l_n(T) - l_n(-T) \leq c$ for every $n \in \mathbb{N}$;
(iii) there is a continuous increasing function $\eta: [0, \infty) \to [0, \infty)$, $\eta(0) = 0$ and an increasing function $K: [-T, T] \to \mathbb{R}$ which is left-continuous on $(-T, T]$, right-continuous at $-T$ and such that

$$k_n(v_n^{-1}(s_2)) - k_n(v_n^{-1}(s_1)) \leq \eta(K(s_2) - K(s_1)) \quad \text{for every } n \in \mathbb{N},$$

$$-T \leq s_1 < s_2 \leq T;$$

(iv) for every $n \in \mathbb{N}$ let us denote by $H_n$ the prolongation of the function $F_n$ along the function $v_n$; then $H_n(x, t) \to H(x, t)$ for every $(x, t) \in G$.

3.3. Proposition. Let a sequence $F_n \in \Phi(G, k_n, l_n, \omega)$, $n \in \mathbb{N}$ converge under substitution to a pair $(H, v)$. Then there are functions $\kappa, \lambda$ such that $H \in \Phi(G, \kappa, \lambda, \omega)$ and

$$\kappa(s_2) - \kappa(s_1) \leq \eta(K(s_2) - K(s_1)) \quad \text{if } -T \leq s_1 < s_2 \leq T,$$

$$\lambda(T) - \lambda(-T) \leq c.$$

Proof. Denote $\kappa_n(s) = k_n(v_n^{-1}(s)) - k_n(-T)$, $\lambda_n(s) = l_n(v_n^{-1}(s)) - l_n(-T)$; then $\kappa_n(-T) = \lambda_n(-T) = 0$ and from (3.6) we get the inequality $\kappa_n(s_2) - \kappa_n(s_1) \leq \eta(K(s_2) - K(s_1))$ for $n \in \mathbb{N}, -T \leq s_1 < s_2 \leq T$.

As was stated in 1.4, this inequality implies that the sequence $(\kappa_n)_n$ contains a subsequence $(\kappa_{n_k})$ which converges uniformly on $[-T, T]$ to a function $\kappa$; the relation (3.7) obviously holds.

Since the functions $\lambda_n$ are nondecreasing and bounded by the constant $c$, by Helly’s Choice Theorem the sequence $(\lambda_{n_k})_k$ contains a subsequence, for simplicity denoted again by $(\lambda_{n_k})$, such that $\lambda_{n_k}(s) \to \lambda(s)$ for every $s \in [-T, T]$. Define $\lambda(s) = \lambda(s^-)$ for $s \in (-T, T]$, $\lambda(-T) = \lambda((-T)^+)$. Obviously $\lambda$ is nondecreasing, $\lambda(T) - \lambda(-T) \leq c$.

Since $H_n \in \Phi(G, \kappa_n, \lambda_n, \omega)$ for every $n \in \mathbb{N}$, we have $|H_n_k(x, s_2) - H_n_k(x, s_1)| \leq \kappa_{n_k}(s_2) - \kappa_{n_k}(s_1)$ for every $k \in \mathbb{N}$ and $-T < s_1 < s_2 < T, x \in \Omega$. Passing to infinity we get the inequality $|H(x, s_2) - H(x, s_1)| \leq \kappa(s_2) - \kappa(s_1)$.

Similarly $|H(x, s_2) - H(x, s_1) - H(y, s_2) + H(y, s_1)| \leq \omega(|x - y|)(\kappa(s_2) - \kappa(s_1))$ provided $x, y \in \Omega, -T < s_1 < s_2 < T$. If the function $\kappa$ is left-continuous at the points $s_1$ and $s_2$ then $\lambda(s_1) = \kappa(s_1), \lambda(s_2) = \kappa(s_2)$, hence the inequality.
3.8) \[ |H(x, s_2) - H(x, s_1) - H(y, s_2) + H(y, s_1)| \leq \omega([x - y]) \left(\lambda(s_2) - \lambda(s_1)\right) \]
holds. Since the functions \(H(x, \cdot), H(y, \cdot), \lambda\) are left-continuous on \((-T, T]\) and right-continuous at \(-T\), we conclude that (3.8) holds for arbitrary \(s_1, s_2, -T \leq s_1 < s_2 \leq T\).

3.4. Proposition. Assume that a sequence \(F_n \in \Phi(G, k_n, l_n, \omega), n \in \mathbb{N}\) converges under substitution to a pair \((H, v)\).

For every \((x, t) \in G\) let us define \(F(x, t) = H(x, v(t))\). Then \(F_n(x, t) \to F(x, t)\) for every \((x, t) \in G\) such that the function \(K \circ v\) is continuous at \(t\) (the notation from Definition 3.2 is used).

Proof. If the function \(K \circ v\) is continuous at \(t\) then \(v\) is continuous at \(t\) and \(K\) is continuous at \(v(t)\).

Let \(\varepsilon > 0\) be given. There is \(\xi > 0\) such that \(\omega(\xi) < \varepsilon\), further there is \(\delta > 0\) such that \(|K(s) - K(v(t))| < \xi\) for every \(s \in [-T, T]\) such that \(|s - v(t)| < \delta\).

There is an integer \(n_0 \in \mathbb{N}\) such that \(|v_n(t) - v(t)| < \delta\) and \(|H_n(x, v(t)) - H(x, v(t))| < \varepsilon\) for every \(n \geq n_0\).

We have the estimate
\[
|F_n(x, t) - F(x, t)| = |H_n(x, v_n(t)) - H(x, v(t))| \leq |H_n(x, v_n(t)) - H_n(x, v(t))| + |H_n(x, v(t)) - H(x, v(t))| < \eta(|K(v_n(t)) - K(v(t))|) + \varepsilon < \eta(\xi) + \varepsilon < 2\varepsilon \quad \text{for} \quad n \geq n_0.
\]

3.5. Proposition. Assume that a sequence \(F_n \in \Phi(G, k_n, l_n, \omega), n \in \mathbb{N}\) converges under substitution to a pair \((H, v)\). Define \(F(x, t) = H(x, v(t))\) for \((x, t) \in G\). If \(F_0: G \to \mathbb{R}^N\) is the reduction of the function \(H\) by \(v\) and \(F^*(x, t) = F(x, t) - F_0(x, t)\) for \((x, t) \in G\), then there is a nondecreasing jump function \(h: [-T, T] \to \mathbb{R}\) such that

\[
|F^*(x, t_2) - F^*(x, t_1)| \leq |h(t_2) - h(t_1)| \quad \text{for} \quad (x, t_1), (x, t_2) \in G
\]
and

\[
h(t_2) - h(t_1) \leq 2\eta(K(v(t_2)) - K(v(t_1))), \quad -T \leq t_1 < t_2 \leq T.
\]

Proof. By Proposition 3.3 the function \(H\) belongs to \(\Phi(G, x, \lambda, \omega)\) where the function \(x\) satisfies (3.7). Then the function \(h(t) = 2 \sum_{-T < s < t \quad v(s) < v(t)} [x(v(s) +) - x(v(s))]\) satisfies (3.10) and the relation (3.9) follows immediately from Proposition 1.21.

Remark. (i) \(h\) is the jump part of the function \(2x \circ v\).

(ii) (3.10) implies that if the function \(K \circ v\) is continuous at \(t\) then \(h\) is as well.

3.6. Theorem. Assume that a sequence \(F_n \in \Phi(G, k_n, l_n, \omega), n \in \mathbb{N}\) converges under substitution to a pair \((H, v)\). Let \(F_0: G \to \mathbb{R}^N\) be the reduction of the function
There is a continuous increasing function \( \eta : [0, \infty) \to [0, \infty) \), \( \eta(0) = 0 \) and an increasing function \( h_0 : [-T, T] \to \mathbb{R} \) which is left-continuous on \((-T, T]\) and such that

\[
\lim_{n \to \infty} \sup [k_n(t_2) - k_n(t_1)] \leq \eta(h_0(t_2) - h_0(t_1))
\]

if the function \( h_0 \) is continuous at \( t_1 \) and \( t_2 \), \( -T \leq t_1 < t_2 \leq T \).

Proof. We use the notation from Definition 3.2; let us define \( F(x, t) = H(x, v(t)) \), \( F^*(x, t) = F(x, t) - F_0(x, t) \) and \( h_0(t) = K(v(t)) + 2\alpha(v(t)) \), where \( \alpha \) has the same meaning as in Prop. 3.3.

If the function \( h_0 \) is continuous at \( t_1 \) and \( t_2 \), \( -T \leq t_1 < t_2 \leq T \), then the function \( v \) is continuous at \( t_1 \) and \( t_2 \) and the functions \( K \) and \( \alpha \) are continuous at \( v(t_1) \) and \( v(t_2) \). As in the proof of Prop. 3.3 let us put \( \alpha_n(s) = k_n(v_n^{-1}(s)) - k_n(-T) \) for \( b \in \mathbb{N} \) and \( s \in [-T, T] \). Then \( k_n(t_i) = \alpha_n(v_n(t_i)) + k_n(-T) \) (\( i = 1, 2 \)) and by (3.6) we have

\[
\lim_{n \to \infty} \sup [k_n(t_2) - k_n(t_1)] = \lim_{n \to \infty} \sup [\alpha_n(v_n(t_2)) - \alpha_n(v_n(t_1))] \leq \lim_{n \to \infty} \eta(K(v_n(t_2)) - K(v_n(t_1))) = \eta(K(v(t_2)) - K(v(t_1))) \leq \eta(h_0(t_2) - h_0(t_1)).
\]

If \( h \) has the same meaning as in the proof of Prop. 3.5 (i.e., \( h \) is the jump part of \( 2\alpha \circ v \)), then by Prop. 3.5 we have

\[
|F^*(x, t_2) - F^*(x, t_1)| \leq h(t_2) - h(t_1) \leq h^*(t_2) - h^*(t_1) \quad \text{for} \quad x \in \Omega,
\]

\(-T < t_1 < t_2 < T\), where \( h^* \) is the jump part of the function \( h_0 \). This completes the proof of (3.11).

The condition (3.3) follows from Propositions 3.4 and 3.5.

Let a pair \((x_0, t_0) \in R = R_{(H, v)}\) be given such that \( h_0(t_0^+) > h_0(t_0) \). Let \( \varepsilon > 0 \) be given.

In case that \( v(t_0) = v(t_0^+) \), let us find such \( \Delta > 0 \) that \( \eta(K(v(t_0)) + \Delta) - K(v(t_0)) + \Delta) < \varepsilon/24 \). There is such an integer \( n_1 \) that

\[
|H_n(x_0, v(t_0)) - H(x_0, v(t_0))| < \varepsilon/8 \quad \text{and} \quad |H_n(x_0, v(t_0)) + \Delta) - H(x_0, v(t_0)) + \Delta| < \varepsilon/24 \quad \text{for every} \quad n \geq n_1.
\]

Then

\[
|H_n(x_0, v(t_0)) + \Delta) - H(x_0, v(t_0)) + \Delta| \leq |H_n(x_0, v(t_0)) + \Delta) - H(x_0, v(t_0)) + \Delta| + 2\eta(K(v(t_0)) + \Delta) - K(v(t_0)) + \Delta) < \varepsilon/8.
\]

In case that \( v(t_0) < v(t_0^+) \), by the definition of \( R_{(H, v)} \) in 1.18 there is \( \sigma > 0 \) and a solution \( y_0 \) of the equation

\[
\frac{dy}{d\tau} = DH(y, t)
\]
on \([v(t_0), v(t_0 + \sigma)]\) such that \(y_0(v(t_0)) = x_0\). By Corollary 2.7 there is \(n_1 \in N\) and \(\alpha > 0\) such that, if \(\sigma' \in (0, \sigma]\) and \(y\) is a solution of the equation

\[
(3.14)_n \quad \frac{dy}{dt} = DH_n(y, t) \tag{3.14}_n
\]
on the interval \([v(t_0), v(t_0 + \sigma')]\) for some \(n \geq n_1\), and if \(|y(v(t_0)) - y_0(v(t_0))| < \alpha\), then

\[
(3.15) \quad |y(s) - y_0(s)| < \varepsilon/4 \quad \text{for} \quad s \in [v(t_0), v(t_0 + \sigma')] \tag{3.15}
\]

There is \(r \in (0, \varepsilon/2)\) such that \(\alpha(r) c < \varepsilon/4\); in case \(v(t_0) < v(t_0 +)\) let us assume that also \(r \leq \alpha\). There is \(\varrho > 0\) such that

\[
\eta(K(v(t_0) + \varrho) - K(v(t_0) + \varrho)) < r/2, \quad \eta(K(v(t_0)) - K(v(t_0) - \varrho)) < r/2.
\]

There is such \(\delta \in (0, r/2)\) that the function \(v\) is continuous at the points \(t_0 - \delta\), \(t_0 + \delta\) and \(v(t_0 + \delta) - v(t_0 +) < \varrho/2\), \(v(t_0) - v(t_0 - \delta) < \varrho/2\).

Let \(\delta' \in (0, \delta)\) be given. Find \(\delta'' \in (0, \delta')\) such that the function \(v\) is continuous at \(t_0 - \delta'', t_0 + \delta''\).

Then \(v_n(t_0 - \delta'') \to v(t_0 - \delta'')\), \(v_n(t_0 + \delta'') \to v(t_0 + \delta'')\); since \(v(t_0 + \delta'') > v(t_0 + \delta''/2) > v(t_0) > v(t_0 - \delta'')\), there is an integer \(n_2 \geq n_1\) such that \(v_n(t_0 + \delta'') > v(t_0 + \delta''/2) > v(t_0) > v_n(t_0 - \delta'')\) for every \(n \geq n_2\). Consequently \([v(t_0), v(t_0 +)] \subset (v_n(t_0 - \delta''), v_n(t_0 + \delta''))\) for \(n \geq n_2\). Since \(v_n(t_0 - \delta'') \to v(t_0 - \delta'')\) and \(v_n(t_0 + \delta'') \to v(t_0 + \delta'')\), there is \(n_0 \geq n_2\) such that \(v_n(t_0 - \delta') - v(t_0 - \delta') > \varrho/2\) and \(v(t_0 + \delta') - v(t_0 + \delta') < \varrho/2\). For \(n \geq n_0\) we have \(0 < K(v(t_0)) - K(v(t_0) - \delta') < \varrho/2\) and \(K(v(t_0)) - K(v(t_0) + \delta') < \varrho/2\); hence

\[
(3.16) \quad \eta(K(v(t_0)) - K(v_n(t_0) - \delta'))) \leq \eta(K(v(t_0)) - K(v(t_0) - \varrho)) < r/2.
\]

Similarly it can be proved that

\[
(3.17) \quad \eta(K(v_n(t_0 + \delta')) - K(v(t_0 + \delta')))) \leq \eta(K(v(t_0) + \varrho) - K(v(t_0) + \varrho)) < r/2
\]

for every \(n \geq n_0\).

Let \(x_n\) be a solution of \((3.1)_n\) on \([t_0 - \delta', t_0 + \delta']\) for \(n \geq n_0\) such that \(|x_n(t_0 - \delta') - x_0| \leq \delta\).

If we define \(y_n(t) = x_n(v_n^{-1}(t))\) for \(t \in [v_n(t_0 - \delta'), v_n(t_0 + \delta')]\), then by Theorem 1.17 the function \(y_n\) is a solution of \((3.14)_n\) on \([v_n(t_0 - \delta'), v_n(t_0 + \delta')]\).

Denote \(y_n = y_n(v(t_0))\), \(n \geq n_0\). Then

\[
(3.18) \quad |y_n - x_0| = |y_n(v(t_0)) - x_0| \leq |y_n(v(t_0)) - y_n(v_n(t_0 - \delta'))| + \\
+ |x_n(t_0 - \delta) - x_0| \leq \eta(K(v(t_0)) - K(v_n(t_0 - \delta'))) + \delta \leq r.
\]

a) Assume that \(v(t_0) = v(t_0 +)\). Using Lemma 2.8 in [S1] for functions \(y_n, H_n\) and

\[
y_n(v(t_0) +) - y_n(v(t_0)) = y_n(s +) - y_n(s) = \\
= H_n(y_n(s), s +) - H_n(y_n(s), s) = H_n(y_n, v(t_0) +) - H_n(y_n, v(t_0)).
\]

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Lemma 2.6 in [Si] implies that

$$|y_n(s_2) - y_n(s_1)| \leq \kappa_n(s_2) - \kappa_n(s_1) \leq \eta(K(s_2) - K(s_1))$$

holds for $s_1 < s_2$; hence for $s_1 \to v(t_0) +$ and $s_2 = v(t_0 + \delta')$ we get the inequality

$$|y_n(v_n(t_0 + \delta')) - y_n(v(t_0) + ))| \leq \eta(K(v_n(t_0 + \delta')) - K(v(t_0) + )) ;$$

similarly for $s_1 = v(t_0 - \delta')$ and $s_2 = v(t_0)$ we have

$$|y_n(t_0) - y_n(v_n(t_0 - \delta'))| \leq \eta(K(v(t_0)) - K(v_n(t_0 - \delta'))).$$

From (3.12), (3.16), (3.17) and (3.18) we get the inequality

$$|x_n(t_0 + \delta') - x_n(t_0 - \delta') - [F_0(x_0, t_0) + F_0(x_0, t_0)]| =$$

$$= |y_n(v_n(t_0 + \delta')) - y_n(v_n(t_0 - \delta')) - [H_n(x_0, v(t_0)) + H_n(x_0, v(t_0))]| =$$

$$= |[y_n(v_n(t_0 + \delta')) - y_n(v(t_0) + )) + [H_n(v_n, v(t_0) + ) - H_n(v_n, v(t_0))]| =$$

$$\leq |y_n(v_n(t_0 + \delta')) - y_n(v(t_0) + ))| + |y_n(v(t_0)) - y_n(v_n(t_0 - \delta'))| +$$

$$+ |H_n(v_n, v(t_0) + ) - H_n(v_n, v(t_0)) - H_n(x_0, v(t_0) + ) + H_n(x_0, v(t_0))| +$$

$$+ |H_n(x_0, v(t_0) + ) - H_n(x_0, v(t_0))| + |H_n(x_0, v(t_0)) - H_n(x_0, v(t_0))| =$$

$$\leq \eta|K(v_n(t_0 + \delta')) - K(v(t_0) + )) + \eta(K(v(t_0)) - K(v_n(t_0 - \delta'))| +$$

$$+ \alpha|y_n - x_0|(\lambda_n(v(t_0) + ) - \lambda_n(v(t_0))) + \epsilon/8 + \epsilon/8 <$$

$$< r/2 + r/2 + \alpha(r)c + \epsilon/4 < \epsilon .$$

b) Assume that $v(t_0) < v(t_0 + )$. For $n \geq n_0$ the function $y_n$ is defined on $[v(t_0), v(t_0 + \delta'/2)]$ and according to (3.18) the inequality $|y_n(v(t_0)) - x_0| < r \leq \alpha$ holds; then (3.15) is satisfied for $y = y_n$ and $\sigma' = \delta'/2$. We have the inequality

$$|x_n(t_0 + \delta') - x_n(t_0 - \delta') - [F(x_0, t_0) + F(x_0, t_0)]| =$$

$$= |[y_n(v_n(t_0 + \delta')) - y_n(v_n(t_0 - \delta'))] - [y_0(v(t_0) + ) - y_0(v(t_0))]| \leq$$

$$\leq |y_n(v_n(t_0 + \delta')) - y_n(v(t_0) + ))| + |y_0(v(t_0) + ) - y_0(v(t_0) + ))| +$$

$$+ |y_0(v(t_0)) - y_n(v(t_0))| + |y_n(v(t_0)) - y_n(v_n(t_0 - \delta'))| \leq$$

$$\leq \eta|K(v_n(t_0 + \delta')) - K(v(t_0) + )) + \eta(K(v(t_0)) - K(v_n(t_0 - \delta'))| +$$

$$+ 2|y_n - y_0| < r/2 + r/2 + 2 \cdot \epsilon/4 < \epsilon .$$

Consequently, the condition (3.4) is verified.

**Remark.** Taking into account that (3.11) is a certain "generalization" of the condition (3.2) and the function $F_0$ need not belong to $\mathcal{F}(G, h_0, \omega)$, we can say that the sequence $(F_n)$ in Theorem 3.6 "converges R-emphatically to $F_0$" in a little more general setting.
3.7. Lemma. Let a sequence of functions \( H_n \in \Phi(G, x_n, \lambda_n, \omega), \ n \in \mathbb{N} \) be given such that

\[
(3.19) \quad \text{there is such } c > 0 \text{ that } \lambda_n(T) - \lambda_n(-T) \leq c, \ \ n \in \mathbb{N};
\]

\[
(3.20) \quad \text{there is a continuous increasing function } \eta: [0, \infty) \to [0, \infty), \ \eta(0) = 0,
\]

and an increasing function \( K: [-T, T] \to \mathbb{R} \) which is left-continuous on \((-T, T]\), right-continuous at \(-T\) and such that \( \lambda_n(s_2) - \lambda_n(s_1) \leq \eta(K(s_2) - K(s_1)) \) for every \( n \in \mathbb{N}, -T \leq s_1 < s_2 \leq T; \)

\[
(3.21) \quad \text{there is such } \sigma \in (-T, T) \text{ that } H_n(x, \sigma) = 0 \text{ for every } x \in \Omega, \ n \in \mathbb{N}.
\]

Then \( (H_n)_{n=1}^{\infty} \) contains a pointwise convergent subsequence.

Proof. For every \( n \in \mathbb{N} \) let us define a function \( \hat{H}_n: \Omega \times [K(-T), K(T)] \to \mathbb{R}^N \) in the following way: \( \hat{H}_n(x, \tau) = H_n(x, t) \) for every \( x \in \Omega \) and \( \tau \in (K(-T), K(T)) \) having the form \( \tau = K(t) \). If \( t \in (-T, T) \) is such a point that \( K(t) < K(t^+) \) then \( \hat{H}_n(x, K(t^+)) = H_n(x, t^+) \) and the function \( \hat{H}_n(x, \cdot) \) is defined linearly on \((K(t), K(t^+)) \) (in terms of the notions from [F2], the function \( \hat{H}_n(x, \cdot) \) is the linear prolongation of the function \( H_n(x, \cdot) \) along the function \( K \).

By [F2], Prop. 1.22 there is a continuous concave increasing function \( \hat{f}: [0, \infty) \to [0, \infty) \) where \( \gamma = K(T) - K(-T) \), such that \( \hat{f}(0) = 0 \) and \( \hat{f}(r) \leq \hat{f}(r) \) for every \( r \in [0, \gamma] \). Then for every \( n \in \mathbb{N}, x \in \Omega \) the inequality

\[
|H_n(x, t_2) - H_n(x, t_1)| \leq \lambda_n(t_2) - \lambda_n(t_1) \leq \hat{f}(K(t_2) - K(t_1)), \ -T < t_1 < t_2 < T
\]

holds. From [F2], Prop. 2.9 it follows that

\[
(3.22) \quad |\hat{H}_n(x, \tau_2) - \hat{H}_n(x, \tau_1)| \leq \hat{f}(\tau_2 - \tau_1)
\]

for every \( x \in \Omega, K(-T) < \tau_1 < \tau_2 < K(T), \ n \in \mathbb{N}. \)

The inequality (3.22) implies that the limits \( \hat{H}_n(x, K(-T)^+), \hat{H}_n(x, K(T)^-) \) exist for every \( x \in \Omega, \ n \in \mathbb{N}. \) Let us define \( \hat{H}_n(x, K(-T)) = \lim_{\tau \to K(-T)^+} \hat{H}_n(x, \tau), \)

\( \hat{H}_n(x, K(T)) = \lim_{\tau \to K(T)^-} \hat{H}_n(x, \tau). \) Then the inequality (3.22) holds if \( K(-T) \leq \tau_1 < \tau_2 \leq K(T). \)

Let \( t \in (-T, T) \) be given, denote \( \tau = K(t) \). For every \( x, y \in \Omega, \ n \in \mathbb{N} \) we have

\[
|\hat{H}_n(x, \tau) - \hat{H}_n(y, \tau)| = |H_n(x, t) - H_n(y, t)| =
\]

\[
= |H_n(x, t) - H_n(x, \sigma) - H_n(y, t) + H_n(y, \sigma)| \leq \omega(|x - y|) |\lambda_n(t) - \lambda_n(\sigma)| \leq
\]

\[
\leq \omega(|x - y|) (\lambda_n(T) - \lambda_n(-T)) \leq \omega(|x - y|) c.
\]

If \( K(t_0) < K(t_0^+) \), then passing to the limit with \( t \to t_0^+ \) we get the inequality

\[
(3.23) \quad |\hat{H}_n(x, \tau) - \hat{H}_n(y, \tau)| \leq \omega(|x - y|) c
\]

also for \( \tau = K(t_0^+) \). Since (3.23) holds for \( \tau = K(t_0) \) and for \( \tau = K(t_0^+) \) and the
function \( \bar{A}(x, \cdot) \) is linear on the interval \([K(t_0), K(t_0 +)]\), the inequality (3.23) holds for every \( \tau \in [K(t_0), K(t_0 +)] \). Consequently, (3.23) is valid for every \( \tau \in [K(-T), K(T)] \), \( x \in \Omega \), \( n \in \mathbb{N} \). From (3.22) and (3.23) we get

\[
|\bar{A}(x, \tau_2) - \bar{A}(x, \tau_1)| \leq |\bar{A}(x, \tau_2) - \bar{A}(x, \tau_1)| + \\
+ |\bar{A}(y, \tau_1) - \bar{A}(y, \tau_1)| \leq h(\tau_2 - \tau_1) + o(|x - y|) \quad \text{for} \quad x, y \in \Omega \,, \quad n \in \mathbb{N} \,.
\]

hence the functions \( \bar{A} \) are equicontinuous on \( \Omega \times [K(-T), K(T)] \). By (3.21), (3.22) we have \( |\bar{A}(x, \tau)| = |\bar{A}(x, \tau) - \bar{A}(x, K(\sigma))| \leq h(|\tau - K(\sigma)|) \leq h(K(T) - K(-T)) \), hence the functions \( \bar{A} \) are bounded. It follows from the Arzelà-Ascoli Theorem that for every compact subset \( A \) of \( \Omega \times [K(-T), K(T)] \) using the diagonalization we can find a subsequence \( (\bar{A}(n))_{n=1}^\infty \) which converges pointwise to a function \( \bar{A}: \Omega \times [K(-T), K(T)] \rightarrow \mathbb{R}^N \). If we define \( H(x, t) = \bar{A}(x, K(t)) \) for every \( (x, t) \in G \), then \( H_n(x, t) = \bar{A}_n(x, K(t)) \rightarrow \bar{A}(x, K(t)) = H(x, t) \).

**3.8. Lemma.** Let functions \( F_0, \bar{F}: G \rightarrow \mathbb{R}^N \) be given such that

(i) there is a nondecreasing left-continuous function \( h: [-T, T] \rightarrow \mathbb{R} \) which has zero continuous part, such that

\[
|F_0(x, t_2) - \bar{F}(x, t_2) - F_0(x, t_1) + \bar{F}(x, t_1)| \leq h(t_2) - h(t_1)
\]

for every \( x \in \Omega \), \(-T < t_1 < t_2 < T \);

(ii) there is a set \( \bar{R} \subset G \) such that for every \((x, t) \in \bar{R} \) the identity \( F_0(x, t) - \bar{F}(x, t) = F(x, t) - \bar{F}(x, t) \) holds.

If \( x: [\alpha, \beta] \rightarrow \mathbb{R}^N \), \( [\alpha, \beta] \subset (-T, T) \) is such a function that \((x(t), t) \in \bar{R} \) for every \( t \in [\alpha, \beta] \), then

\[
\int_{t_1}^{t_2} D F_0(x(\tau), t) = \int_{t_1}^{t_2} D \bar{F}(x(\tau), t)
\]

for every \( t_1, t_2 \in [\alpha, \beta] \) provided at least one of the integrals exists.

**Proof.** Let us denote \( N(x, t) = F_0(x, t) - \bar{F}(x, t) \), \((x, t) \in G \). Then

\[
|N(x, t_2) - N(x, t_1)| \leq h(t_2) - h(t_1) \quad \text{for every} \quad x \in \Omega \,, \quad -T < t_1 < t_2 < T ;
\]

(3.24)

\[
N(x, t+) = N(x, t) = 0 \quad \text{for every} \quad (x, t) \in \bar{R} .
\]

Assume that \( \alpha \leq t_1 < t_2 \leq \beta \).

By Lemma 1.6 we have

\[
\int_{t_1}^{t_2} D N(x(\tau), t) = \sum_{t_1 \leq s < t_2} [N(x(s), s+) - N(x(s), s)].
\]
Since \((x(s), s) \in \mathcal{R}\) for every \(s \in [t_1, t_2]\), (3.25) implies that 
\[N(x(s), s+) - N(x(s), s) = 0.\]
Consequently
\[
\int_{t_1}^{t_2} D[F_0(x(\tau), \tau) - F(x(\tau), \tau)] = \int_{t_1}^{t_2} D N(x(\tau), \tau) = 0.
\]
The rest of the Lemma follows from [S1], Th. 1.6.

3.9. Theorem. Assume that a sequence of functions \(F_n \in \mathcal{F}(G, h_n, \omega), n \in \mathbb{N}\) converges \(R\)-emphatically to a function \(F_0: G \to \mathbb{R}^N\). Then

(i) there is a subsequence \((F_{n_k})_{k=1}^{\infty}\) which converges under substitution to a pair \((H, v)\);

(ii) if we denote by \(\bar{F}\) the reduction of the function \(H\) by the function \(v\), then for every \(x \in \Omega\) the continuous part of the function \(F_0(x, \cdot) - \bar{F}(x, \cdot)\) is constant and for every \((x, t) \in R \cap R_{(H, v)}\) the identity \(F_0(x, t+) - F_0(x, t) = F(x, t+) - F(x, t)\) holds;

(iii) let \([\alpha, \beta] \subset (-T, T)\); a function \(x: [\alpha, \beta] \to \mathbb{R}^N\) such that \((x(t), t) \in R \cap R_{(H, v)}\) for every \(t \in [\alpha, \beta]\) is a solution of the equation (3.5), if and only if it is a solution of the equation

\[
(3.26) \quad \frac{dx}{d\tau} = DF(x, t)
\]
on the interval \([\alpha, \beta]\).

Proof. (i) By [F2], Th. 1.21 there is a subsequence \((h_{n_k})_{k=1}^{\infty}\) for which there exists a sequence of continuous increasing functions \((v_k)_{k=1}^{\infty} \subset \Lambda\) and an increasing function \(v \in V^{-}\) such that \(v_k(t) \to v(t)\) for every \(t \in (-T, T)\) at which \(v\) is continuous, and the functions \(h_{n_k} \circ v_k^{-1}\) are equiregulated.

Since the functions \(h_{n_k} \circ v_k^{-1}\) are equiregulated, by the property 1.4 there is an increasing continuous function \(\eta: [0, \infty) \to [0, \infty), \eta(0) = 0\) and an increasing function \(K: [-T, T] \to \mathbb{R}\) which is left-continuous on \((-T, T]\) and right-continuous at \(-T\) so that

\[
h_{n_k}(v_k^{-1}(s_2)) - h_{n_k}(v_k^{-1}(s_1)) \leq \eta(K(s_2) - K(s_1)) \quad \text{for every} \quad k \in \mathbb{N},
\]
\(-T \leq s_1 < s_2 \leq T.

Then the inequality (3.6) is satisfied provided \(k_n, v_n\) are replaced by \(h_{n_k}, v_k\).

Fix such a point \(t_0 \in (-T, T)\) that the functions \(h_{n_k}, v_k\) are continuous at \(t_0\) and \(K\) is continuous at \(v(t_0)\). Let \(H_k: G \to \mathbb{R}^N\) be the prolongation of the function \(F_{n_k}\) along \(v_k\), denote \(H'_k(x, \tau) = H'_k(x, \tau) - H_k(x, v(t_0))\) for every \((x, \tau) \in G\).

By Lemma 3.7 there is a subsequence of \((H'_k)\) which for simplicity will be denoted again by \((H'_k)\), such that \(H'(x, \tau) \to H'(x, \tau)\) for every \((x, \tau) \in G\). Define \(H(x, \tau) = H'(x, \tau) + \lim_{n \to \infty} F_n(x, t_0)\).

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By (3.3) for every $\varepsilon > 0$ there is such an integer $k_0$ that $|F_{n_k}(x, t_0) - \lim_{n \to \infty} F_n(x, t_0)| < \varepsilon$ for every $k \geq k_0$. For $k \geq k_0$ we have $|H_k(x, \nu(t_0)) - \lim_{n \to \infty} F_n(x, t_0)| = |F_{n_k}(x, v_k^{-1}(\nu(t_0))) - \lim_{n \to \infty} F_n(x, t_0)| \leq |F_{n_k}(x, v_k^{-1}(\nu(t_0))) - F_{n_k}(x, t_0)| + |F_{n_k}(x, t_0) - \lim_{n \to \infty} F_n(x, t_0)| < \varepsilon + h_{n_k}(v_k^{-1}(\nu(t_0))) - h_{n_k}(t_0) \leq \varepsilon + \eta(|K(\nu(t_0)) - K(\nu(t_0))|).

Since $v$ is continuous at $t_0$, we conclude that $v_k(t_0) \to v(t_0)$; further $K(v_k(t_0)) \to K(v(t_0))$ because $K$ is continuous at $v(t_0)$. Consequently $H_k(x, t) \to H(x, t)$, $(x, t) \in G$, and the subsequence $(F_{n_k})$ converges under substitution to $(H, v)$.

(ii) By (3.3) there is a function $F(x, t) = F_0(x, t) + F^*(x, t)$ which is left-continuous in $t$ and such that $F_n(x, t) \to F(x, t)$ for every $x \in \Omega$ and $t \in (-T, T)$ at which the function $h_0$ is continuous. Proposition 3.4 yields that $F(t, x, \nu(t))$, $(x, t) \in G$.

By Proposition 3.5 there is such a jump function $h$ that (3.9), (3.10) hold when $F^*(x, t)$ is replaced by $F^*(x, t) = F(x, t) - F^*(x, t)$. Then

$$
[F_0(x, t_2) - F(x, t_2)] - [F_0(x, t_1) - F(x, t_1)] = [F^*(x, t_2) - F^*(x, t_1)] - [F^*(x, t_2) - F^*(x, t_1)] \\
\leq [h^*(t_2) - h^*(t_1)] + [h(t_2) - h(t_1)] = h(t_2) - h(t_1)
$$

where $h(t) = h^*(t) + h(t)$.

If $(x, t) \in R$ then the value $F_0(x, t^+) - F_0(x, t)$ is evaluated by (3.4). Since the subsequence $(F_{n_k})$ converges $R$-emphatically to $F_0$ and $R_{(H, v)}$-emphatically to $F$, we have

$$F_0(x, t^+) - F_0(x, t) = F(x, t^+) - F^*(x, t) \quad \text{for every } (x, t) \in R \cap R_{(H, v)}.
$$

Part (iii) is an evident consequence of Lemma 3.8.

### 3.10. Theorem

Assume that a sequence $F_n \in \Phi(G, k_n, l_n, \omega)$, $n \in \mathbb{N}$ converges under substitution to a pair $(H, v)$, let $F_0: G \to \mathbb{R}^N$ be the reduction of the function $H$ by $v$. Assume that the function $K \circ v$ is continuous at $\alpha, \beta$, $-T < \alpha < \beta < T$.

(i) If $x_n$ is a solution of the equation (3.1) on $[\alpha, \beta]$ for every $n \in \mathbb{N}$ and if the set $\{x_n(x); n \in \mathbb{N}\}$ is bounded, then there is a function $x_0: [\alpha, \beta] \to \mathbb{R}^N$ with bounded variation and left-continuous on $(\alpha, \beta)$, and a subsequence $(x_{n_k})_{k=1}^\infty$ such that $x_{n_k}(t) \to x_0(t)$ for every $t \in [\alpha, \beta]$ at which the function $K \circ v$ is continuous (notation from Def. 3.2 is used).

(ii) Assume that $x_0(\alpha) \in \Omega$. Then either the function $x_0$ is a solution of the generalized differential equation with a substitution

$$x(t) = y(\nu(t)), \quad \frac{dy}{dt'} = DH(y, t')
$$

on the interval $[\alpha, \beta]$, or there is such $\beta' \in (\alpha, \beta]$ that $x_0$ is a maximal solution
on \([\alpha, \beta')\) or \([\alpha, \beta']\), or there is such \(\beta'' \in (\alpha, \beta)\) that \(x_0\) is a solution of (3.27) on \([\alpha, \beta'']\) and disappears at \(\beta''\).

(iii) If \((x_0(t), t) \in R(H, v)\) for every \(t \in [\alpha, \beta]\), then the function \(x_0\) is a solution of (3.27) on \([\alpha, \beta]\), as well as of the equation (3.5).

Proof. (i) By Corollary 2.7 in [S1] the function \(x_n\) has bounded variation on \([\alpha, \beta]\) for every \(n \in \mathbb{N}\) and

\[
\var^\beta x_n \leq k_n(\beta) - k_n(\alpha) \leq \eta(K(v_n(\beta)) - K(v_n(\alpha))) \leq \eta(K(T) - K(-T))
\]

for every \(n \in \mathbb{N}\). By Helly's Choice Theorem there is a function \(x: [\alpha, \beta] \to \mathbb{R}^N\) of bounded variation and a subsequence \((x_{n_k})_{k=1}^{\infty}\) such that \(x_{n_k}(t) \to x(t)\) for every \(t \in [\alpha, \beta]\).

Since \(|x_{n_k}(t_2) - x_{n_k}(t_1)| \leq k_{n_k}(t_2) - k_{n_k}(t_1) \leq \eta(K(v_{n_k}(t_2)) - K(v_{n_k}(t_1)))\) for every \(k \in \mathbb{N}\), \(\alpha \leq t_1 < t_2 \leq \beta\), passing to the limit with \(k \to \infty\) we get the inequality

\[
|x(t_2) - x(t_1)| \leq \eta(K(v(t_2)) - K(v(t_1)))
\]

for every \(t_1, t_2\) at which the function \(K \circ v\) is continuous, \(\alpha \leq t_1 < t_2 \leq \beta\).

Let us define \(x_0(\alpha) = x(\alpha), x_0(t) = x(t -)\) for \(t \in (\alpha, \beta]\). Then the function \(x_0\) has bounded variation and is left-continuous on \((\alpha, \beta]\). Since the function \(x\) has one-sided limits on \([\alpha, \beta]\), the inequality (3.28) yields

\[
\begin{align*}
|x(\alpha^+) - x(\alpha)| &\leq \eta(K(v(\alpha^+) - K(v(\alpha))) = 0; \\
|x(t) - x(t^-)| &\leq \eta(K(v(t)) - K(v(t^-))) = 0
\end{align*}
\]

for every \(t \in (\alpha, \beta]\) at which \(K \circ v\) is continuous.

If the function \(K \circ v\) is continuous at \(t \in [\alpha, \beta]\) then the function \(x\) is continuous at \(t\), which implies that \(x_0(t) = x(t) = \lim_{k \to \infty} x_{n_k}(t)\).

(ii) For every \(t \in [\alpha, \beta]\), \(n \in \mathbb{N}\) we have the estimate \(|x_n(t)| \leq |x_n(\alpha)| + \eta(K(T) - K(-T))\), hence there is \(d > 0\) such that

\[
|x_n(t)| \leq d \quad \text{for every} \quad t \in [\alpha, \beta]\,
\]

For every \(k \in \mathbb{N}\) let us define \(y_k(\tau) = x_{n_k}(v_{n_k}^{-1}(\tau)), \tau \in [v_{n_k}(\alpha), v_{n_k}(\beta)]\). By Theorem 1.17 the function \(y_k\) is a solution of the equation (3.14) on \([v_{n_k}(\alpha), v_{n_k}(\beta)]\).

Since the function \(v\) is continuous at \(\alpha, \beta\), we have \(v_{n_k}(\alpha) \to v(\alpha), v_{n_k}(\beta) \to v(\beta)\). Hence for every \([\gamma, \delta] \subset (v(\alpha), v(\beta))\) there is such \(k_0 \in \mathbb{N}\) that \([\gamma, \delta] \subset [v_{n_k}(\alpha), v_{n_k}(\beta)]\) for every \(k \geq k_0\). By Theorem 2.5 the sequence \(y_k\) contains a subsequence which is uniformly convergent on \([\gamma, \delta]\). By a diagonalization process we can find a function \(y_0: (v(\alpha), v(\beta)) \to \mathbb{R}^N\) and a subsequence of \((y_k)\) — which will be denoted again by \((y_k)\) — so that \(y_k \Rightarrow y_0\) on \([\gamma, \delta]\) for every \([\gamma, \delta] \subset (v(\alpha), v(\beta))\).

From (3.2) and Lemma 2.6 in [S1] it follows that

\[
|y_k(s_2) - y_k(s_1)| \leq \eta(K(s_2) - K(s_1)), \quad v_{n_k}(\alpha) \leq s_1 < s_2 \leq v_{n_k}(\beta), \quad k \in \mathbb{N};
\]

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then
\[ (3.31) \quad |y_0(s_2) - y_0(s_1)| \leq \eta(K(s_2) - K(s_1)), \quad v(\alpha) < s_1 < s_2 < v(\beta). \]

Let us define \( y_0(v(\alpha)) = y_0(v(\alpha) +), \ y_0(v(\beta)) = y_0(v(\beta) -). \) If the function \( K \circ v \) is continuous at \( t \in (\alpha, \beta) \) then
\[
|x_0(t) - y_0(v(t))| \leq \|x_0(t) - x_n(t)\| + |y_n(v_n(t)) - y_0(v_n(t))| +
+ |y_0(v_n(t)) - y_0(v(t))| \leq |x_0(t) - x_n(t)| +
+ \|y_n - y_0\|_{\mathcal{C}(\mathbb{R}^3)} + \eta(K(v_n(t)) - K(v(t)))
\]
where \( \delta > 0 \) is so small that \( (t - \delta, t + \delta) \subseteq (v(\alpha), v(\beta)). \) The expression at the end of the inequalities tends to zero with \( k \to \infty, \) hence \( x_0(t) = y_0(v(t)) \) for every \( t \in (\alpha, \beta) \) at which the function \( K \circ v \) is continuous. Since the functions \( x_0, y_0, \) \( v \) are left-continuous, the equality \( x_0(t) = y_0(v(t)) \) holds for every \( t \in (\alpha, \beta]. \) The continuity of \( K \circ v \) at \( \alpha \) implies that the functions \( x_0, v \) are right-continuous at \( \alpha \) and \( y_0 \) is right-continuous at \( v(\alpha) \). Hence
\[ (3.32) \quad x_0(t) = y_0(v(t)) \quad \text{for every} \quad t \in [\alpha, \beta]. \]

Since \( x_0(\alpha) = y_0(v(\alpha)) \in \Omega \) and the function \( y_0 \) is right-continuous at \( v(\alpha), \) there is such \( \delta > 0 \) that \( y_0(\tau) \in \Omega \) for every \( \tau \in [v(\alpha), v(\alpha) + \delta). \)

If \( y_0(\tau) \in \Omega \) for every \( \tau \in [v(\alpha), v(\beta)], \) by Theorem 2.5 the function \( y_0 \) is a solution of the equation (3.13); then the function \( x_0 = y_0 \circ v \) is a solution of (3.27) on \([\alpha, \beta].\)

Assume that there is such \( \gamma \in (v(\alpha), v(\beta)] \) that \( y_0(\tau) \in \Omega \) for every \( \tau \in [v(\alpha), \gamma) \) and \( y(\gamma) \notin \Omega. \) If \( \gamma = v(\beta') \) for some \( \beta' \in (\alpha, \beta] \) then the function \( x_0 \) is a maximal solution of (3.27) on \([\alpha, \beta'). \) If there is such \( \beta'' \in (\alpha, \beta) \) that \( \gamma = (v(\beta''), v(\beta'' +)) \) then the function \( x_0 \) is a solution of (3.27) on \([\alpha, \beta'') \) and disappears at \( \beta''. \)

Finally, assume that there is such \( \tilde{\gamma} \in (v(\alpha), v(\beta)) \) that \( y_0(\tau) \in \Omega \) for every \( \tau \in [v(\alpha), \tilde{\gamma}], \) but there is no \( s > \tilde{\gamma} \) such that \( y_0(\tau) \in \Omega \) for every \( \tau \in [v(\alpha), s]. \) Let us find such \( \beta' \) that \( \gamma \in (v(\beta), v(\beta +]). \) If \( v(\beta) = v(\beta +) \) then \( x_0 \) is a maximal solution on \([\alpha, \beta]. \) If \( v(\beta) < v(\beta +) \) then \( x_0 \) is a solution on \([\alpha, \beta] \) and disappears at \( \beta. \)

(iii) If \((x_0(t), t) \in R_{H, \delta}, \) for every \( t \in [\alpha, \beta] \) then \( y_0(\tau) \in \Omega \) for every \( \tau \in [v(\alpha), v(\beta)], \) hence the function \( x_0 \) is a solution of (3.27). By Theorem 1.24 the function \( x_0 \) is also a solution of (3.5) on \([\alpha, \beta]. \)

3.11. Theorem. Assume that functions \( F_n \in \Phi(G, k_n, l_n, \omega), \) \( n \in \mathbb{N} \) converge under-substitution to a pair \((H, \psi), \) let \( F_0: G \to \mathbb{R}^N \) be the reduction of the function \( H \) by \( \psi. \) Assume that the function \( K \circ v \) is continuous at \( \alpha, \beta, -T < \alpha < \beta < T. \)

Let \( x_0: [\alpha, \beta] \to \mathbb{R}^N \) be a solution of the equation (3.5) on \([\alpha, \beta] \) which has the uniqueness property (2.10) when (2.7) is replaced by (3.27). Assume that \((x_0(t), t) \in R_{H, \delta}) \) for every \( t \in [\alpha, \beta]. \)

Assume that any solution \( y: [v(\alpha), v(\beta)] \to \mathbb{R}^N \) of the equation (3.13) such that \( y(v(\alpha)) = x_0(\alpha) \) satisfies
(3.33) there is such q > 0 that if \( z \in \mathbb{R}^n \), \( s \in [v(\alpha), v(\beta)] \) and \( |z - y(s)| \leq q \) then 
\[ z \in \Omega . \]

Let a sequence \( (z_n)_{n=1}^{\infty} \subset \Omega \) be given such that \( \lim_{n \to \infty} z_n = x_0(\alpha) \).

Then there is an integer \( n_0 \) such that for every \( n \geq n_0 \) there is a solution \( x_n \) of the

\[ \text{equation (3.1)_n on } [\alpha, \beta] \text{ such that } x_0(\alpha) = z_n, \text{ and } \lim_{n \to \infty} x_n(t) = x_0(t) \text{ for every } \]
\[ t \in [\alpha, \beta] \text{ at with the function } K \circ v \text{ is continuous.} \]

Proof. We will use the notation from Definition 3.2. By Theorem 1.24 the function \( x_0 \) is a solution of (3.27) on \([\alpha, \beta]\), hence there is a solution \( y_0 \) of the equation (3.13) on \([v(\alpha), v(\beta)]\) such that \( y_0(v(t)) = x_0(t) \) for every \( t \in [\alpha, \beta] \).

If \( \delta \in (v(\alpha), v(\beta)] \) and \( y \) is a solution of (3.13) on \([v(\alpha), \delta]\) such that \( y(v(\alpha)) = x_0(\alpha) \), let us find such \( \gamma \in [x, \beta] \) that \( \delta \in [v(\gamma), v(\gamma + \gamma)] \) and define \( x(t) = y(v(t)), t \in [x, \gamma] \).

Then the function \( x \) is a solution of the equation (3.27) on \([x, \gamma]\) and by the uniqueness property (2.10) we obtain that \( x(t) = x_0(t) \) for every \( t \in [x, \gamma] \).

For every \( \tau \in [v(\alpha), \delta] \) let us find such \( \eta \in [\tau, \gamma] \) that \( \tau \in [\tau, \tau + \tau] \). If \( \tau = v(\tau) \) then \( y(\tau) = x(\tau) = x_0(\tau) \). If \( \tau < v(\tau) \) and \( \tau \in (v(\tau), v(\tau + \tau]) \), then the definition of the set \( R_{1(\eta, \epsilon)} \) in 1.18 implies that \( y(\tau) = x_0(\tau) \). We have proved that

\[ (3.34) \text{ if } y: [v(\alpha), \delta] \to \mathbb{R}^n \text{ is a solution of (3.13) such that } y(v(\alpha)) = y_0(v(\alpha)) \text{ then } y(\tau) = y_0(\tau), \tau \in [v(\alpha), \delta]. \]

Since the function \( K \circ v \) is continuous at \( \alpha \), there is such \( d > 0 \) that \( \eta(K(v(\alpha) + d) - K(v(\alpha) - d)) < \epsilon/2 \) (the number \( \epsilon \) is taken from (3.33)). Since \( v_n(\alpha) \to v(\alpha) \) and \( z_n \to x_0(\alpha) \), there is such an integer \( n_0 \) that \( v_n(\alpha) \in [v(\alpha) - d, v(\alpha) + d] \) and \( |z_n - x_0(\alpha)| < \epsilon/2 \) for every \( n \geq n_0 \). Let \( n \geq n_0 \) be fixed.

By \( A \) let us denote the set of all functions \( y \) from \( \mathcal{B}[v(\alpha) - d, v(\alpha) + d] \) satisfying \( |y(\tau) - z_n| \leq \eta(|K(\tau) - K(v_n(\alpha))|) \) for every \( \tau \in [v(\alpha) - d, v(\alpha) + d] \). The set \( A \) is closed in \( \mathcal{B}[v(\alpha) - d, v(\alpha) + d] \) and \( y(v_n(\alpha)) = z_n \) for \( y \in A \). If \( y \in A \) then \( |y(\tau) - y_0(v(\alpha))| \leq |y(\tau) - y(v(\alpha))| + |y(v(\alpha)) - x_0(\alpha)| \leq \eta(|K(\tau) - K(v_n(\alpha))|) + \epsilon/2 \) for every \( \tau \in [v(\alpha) - d, v(\alpha) + d] \), hence \( y(\tau) \in \Omega \) owing to (3.33).

For every \( y \in A \) the function

\[ Ty(\sigma) = z_n + \int_{\nu(\alpha)}^{\sigma} DH_n(y(\tau), \tau), \sigma \in [v(\alpha) - d, v(\alpha) + d] \]

is defined. For \( y \in A \) we have

\[ (3.35) \quad |Ty(\sigma_2) - Ty(\sigma_1)| \leq x_n(\sigma_2) - x_n(\sigma_1) \leq \eta(K(\sigma_2) - K(\sigma_1)) , \]

\[ v(\alpha) - d \leq \sigma_1 < \sigma_2 \leq v(\alpha) + d , \]

consequently the set \( T(A) \) is relatively compact in \( \mathcal{B}[v(\alpha) - d, v(\alpha) + d] \) and \( T(A) \subset A \). According to Lemma 2.1 the operator \( T \) is continuous. By the Schauder-Tikhonov fixed point theorem there is such a function \( y_n \in A \) that \( y_n(\tau) = Ty_n(\tau) \),
\[ \tau \in [v(\alpha) - d, v(\alpha) + d]. \] By Theorem 2.5 there is a subsequence \((y_{n_k})\) which converges uniformly to a function \(y\) on \([v(\alpha) - d, v(\alpha) + d]\). Since \(|y(\tau) - y_0(v(\alpha))| < \varepsilon\), \(n \geq n_0\), we have \(|y(\tau) - y_0(v(\alpha))| \leq \varepsilon\) for every \(\tau \in [v(\alpha) - d, v(\alpha) + d]\); the assumption (3.33) implies that \(y(\tau) \in \Omega\). By Theorem 2.5 the function \(y\) is a solution of (3.13) on \([v(\alpha) - d, v(\alpha) + d]\). From (3.34) it follows that \(y(\tau) = y_0(\tau)\) for every \(\tau \in [v(\alpha), v(\alpha) + d]\), consequently \(y_n \rightharpoonup y_0\) on \([v(\alpha), v(\alpha) + d]\).

Since \(y_n'(v(\alpha) + d) \rightarrow y_0'(v(\alpha) + d)\) and the solution \(y_0\) satisfies the assumptions of Theorem 2.6 on \([v(\alpha) + d, v(\beta)]\), there is such \(n_1 \geq n_0\) that for every \(n \geq n_1\) there is a solution \(y_n\) of (3.14) on the interval \([v(\alpha) + d, v(\beta)]\) such that \(y_n(v(\alpha) + d) = y_n'(v(\alpha) + d)\) and \(y_n \rightharpoonup y_0\) on \([v(\alpha) + d, v(\beta)]\). If we define \(y_n(\tau) = y_n'(\tau)\), \(\tau \in [v(\alpha) - d, v(\alpha) + d]\), then \(y_n\) is a solution of (3.14) on \([v(\alpha), v(\beta)]\), \(y_n(v(\alpha)) = z_n\) and \(y_n \rightharpoonup y_0\) on \([v(\alpha), v(\beta)]\).

Since \(y_0(v(\beta)) \in \Omega\), \(y_n(v(\beta)) \rightharpoonup y_0(v(\beta))\) and the function \(K \circ v\) is continuous at \(\beta\), it can be proved similarly as above that there are such \(d' > 0\) and \(n_2 \geq n_1\) that the solutions \(y_n\) can be continued on \([v(\alpha) - d, v(\beta) + d']\) and \(v_n(\beta) \in [v(\beta) - d', v(\beta) + d']\), \(n \geq n_2\).

For every \(n \geq n_2\) let us define \(x_n(t) = y_n(v_n(t))\), \(t \in [\alpha, \beta]\); by Theorem 1.17 the function \(x_n\) is a solution of the equation (3.1) on \([\alpha, \beta]\), \(x_n(\alpha) = y_n(v_n(\alpha)) = z_n\).

If the function \(K \circ v\) is continuous at \(t \in (\alpha, \beta)\) and \(n \geq n_2\), then there is such \(n' \geq n_2\) that \(v_n(t) \in [v(\alpha), v(\beta)]\), \(n \geq n'\). We have \(|x_n(t) - x_0(t)| = |y_n(v_n(t)) - y_0(v(t))| \leq |y_n(v_n(t)) - y_n(v(t))| + |y_n(v(t)) - y_0(v(t))| \leq \|y_n - y_0\|_{[v(\alpha), v(\beta)]} + \eta(\|K(v_n(t)) - K(v(t))\|)\); the last expression tends to zero with \(n \to \infty\).

Let \(\varepsilon > 0\) be given; there is such \(\delta > 0\) that \(\eta(\delta) < \varepsilon\). Find such \(t \in (\alpha, \beta)\) that the function \(K \circ v\) is continuous at \(t\) and \(K(v(\beta)) - K(v(t)) < \delta/2\). There is such \(n'' \geq n_2\) that
\[
|K(v_n(\beta)) - K(v(\beta))| < \delta/4, \quad |K(v_n(t)) - K(v(t))| < \delta/4
\]
and
\[
|x_n(t) - x_0(t)| < \varepsilon \quad \text{for every} \quad n \geq n''.
\]

Then
\[
|x_n(\beta) - x_0(\beta)| \leq |x_n(\beta) - x_n(t)| + |x_n(t) - x_0(t)| + |x_n(t) - x_0(t)| <
\]
\[
< \eta(K(v_n(\beta)) - K(v_n(t))) + \eta(K(v(\beta) - K(v(t))) + \varepsilon <
\]
\[
< \eta(K(v(\beta)) - K(v(t))) + \delta/2 + \eta(\delta) + \varepsilon < 3\varepsilon.
\]

Consequently \(x_n(t) \to x(\beta)\) for every \(t \in [\alpha, \beta]\) at which the function \(K \circ v\) is continuous.

3.12. Example. Assume that functions \(F \in \mathcal{F}(G, h, \omega), \ g: \Omega \to \mathbb{R}^n\) and \(\Phi_n: [-T, T] \to \mathbb{R}, \ n \in \mathbb{N}\) are given such that
(i) the function \(h\) is continuous on \([-T, T]\) and the function \(g\) is uniformly continuous on \(\Omega\);
(ii) the functions \(\Phi_n, \ n \in \mathbb{N}\) are continuous on \([-T, T]\) and there is such \(c > 0\) that \(\text{var}_{-T}^{T} \Phi_n \leq c\) for every \(n \in \mathbb{N}\);
(iii) for every \( \varepsilon > 0 \) there is such \( \delta > 0 \) that
\[
\text{var}_{-T}^{-T+\delta} \Phi_n + \text{var}_{-T}^{-T-\delta} \Phi_n < \varepsilon \quad \text{for every } n \in \mathbb{N};
\]

(iv) there is a function \( \Phi \in BV[-T, T] \) which is left-continuous on \((-T, T]\), right-continuous at \(-T\) and such that \( \Phi_n(t) \to \Phi(t) \) for every \( t \in [-T, T] \) at which \( \Phi \) is continuous (including \(-T, T\)).

Our aim is to find a limit equation for the sequence of generalized differential equations
\[
(3.36)_n \quad \frac{dx}{dt} = D[F(x, t) + g(x) \Phi_n(t)]
\]
(see also [S2], Example 4.7).

Denote \( c_n = 1 + (1/2T) \text{var}_{-T}^{-T} \Phi_n \) for \( n \in \mathbb{N}, \bar{c} = 1 + (1/2T) c \). By (ii), for any \( n \in \mathbb{N} \) the inequality \( 1 \leq c_n \leq \bar{c} \) holds.

For every \( n \in \mathbb{N} \) let us define functions \( v_n(t) = (1/c_n) \left[ t + T + \text{var}_{-T}^{-T} \Phi_n \right] - T, t \in [-T, T] \). Then \( v_n \in \Lambda \). Since the functions \( v_n, n \in \mathbb{N} \) are increasing and bounded, there is a subsequence \( (v_{n_k}) \) and a nondecreasing function \( v_0: [-T, T] \to \mathbb{R} \) such that \( v_{n_k}(t) \to v_0(t) \) for every \( t \in [-T, T] \). Evidently \( v_0(-T) = -T, v_0(T) = T \).

Let us prove that \( v_0 \) is continuous at the endpoints of \([-T, T]\). For \( \varepsilon > 0 \) given let us find \( \delta \in (0, \varepsilon] \) by the assumption (iii). For any \( t \in (-T, -T+\delta] \) we have
\[
v_0(t) - v_0(-T) = \lim_k \left[ v_{n_k}(t) - v_{n_k}(-T) \right] = \lim_k \left[ \frac{1}{c_{n_k}} \left[ t + T + \text{var}_{-T}^{-T} \Phi_{n_k} \right] \right] \leq (t + T) + \lim sup \text{var}_{-T}^{-T} \Phi_{n_k} < 2\varepsilon;
\]
hence \( v_0 \) is right-continuous at \(-T\). The left-continuity at \( T \) can be proved similarly.

We have
\[
(3.37) \quad v_n(t_2) - v_n(t_1) \geq \frac{1}{c_n} (t_2 - t_1) \geq \frac{1}{\bar{c}} (t_2 - t_1) \quad \text{for } n \in \mathbb{N},
\]
\[
n \in \mathbb{N}, \quad -T \leq t_1 < t_2 \leq T,
\]
and consequently
\[
v_0(t_2) - v_0(t_1) \geq \frac{1}{\bar{c}} (t_2 - t_1) \quad \text{for } -T \leq t_1 < t_2 \leq T.
\]

Let us define \( v(-T) = -T, v(T) = T, v(t) = v_0(t^-) \) for \( t \in (-T, T) \). Then obviously \( v \in V^- \) and \( v_{n_k}(t) \to v(t) \) for every \( t \in [-T, T] \) at which \( v \) is continuous.

Let us define \( \Psi_n(\tau) = \Phi_n(v_n^{-1}(\tau)) \) for \( \tau \in [-T, T], n \in \mathbb{N} \). If \( -T \leq \tau_1 < \tau_2 \leq T \) and \( \tau_1 = v_n(t_1), \tau_2 = v_n(t_2) \) for some \( n \in \mathbb{N} \), then
\[
|\Psi_n(\tau_2) - \Psi_n(\tau_1)| = |\Phi_n(t_2) - \Phi_n(t_1)| \leq \text{var}_{t_1}^{t_2} \Phi_n \leq c_n (v_n(t_2) - v_n(t_1)) = c_n (\tau_2 - \tau_1) \leq \bar{c} (\tau_2 - \tau_1).
\]
Then the sequence $(\Psi_n)$ is equicontinuous; it is also bounded, because $|\Phi_n(t)| \leq |\Phi_n(t_0)| + \text{var}^T \Phi_n$ and the sequence $(\Phi_n(t_0))$ is convergent for some $t_0$.

By the Arzelà-Ascoli Theorem the sequence $(\Psi_n)$ contains a subsequence — which will be denoted again by $(\Psi_n)$ — such that $\Psi_n \rightharpoonup \Psi$ on $[-T, T]$. Evidently $\Psi(v(t)) = \phi(t)$. From (3.37) it follows that the functions $v_n^{-1}$ are Lipschitzian with the constant $1/\varepsilon$, hence they converge uniformly to the function $u$ defined by $u(\tau) = t$ if $v(t) \leq \tau \leq v(t+)$ (see [F2], Def. 1.10 and Prop. 1.11).

By Theorem 1.17 the equation (3.36) has the same solutions as the generalized differential equation with a substitution

$$\tag{3.38} x(t) = y(v_n(t)), \quad \frac{dy}{dt'} = D[F(y, v_n(t')) + g(y) \Psi_n(t')] .$$

Since the function $u$ is continuous, we have $F(x, v_n^{-1}(t)) \to F(x, u(t))$ for every $x \in \Omega, \ t \in [-T, T]$. It is simple to verify that the sequence of functions $F_n(x, t) = F(x, t) + g(x) \phi_n(t)$ converges under substitution to the pair $(H, v)$ where $H(x, t) = F(x, u(t)) + g(x) \Psi(t)$; then the equation with a substitution

$$\tag{3.39} x(t) = y(v(t)), \quad \frac{dy}{dt'} = D[F(y, u(t)) + g(y) \Psi(t)]$$

is a limit equation for the sequence (3.36)$_m$.

In case the functions $\phi_n$ satisfy the condition $\text{var}^{-\delta}_T \phi_n + \text{var}^T \phi_n \to 0$ for every $\delta > 0$, the function $\Psi$ will be constant on $[T, v(0)]$ and on $(v(0+), T]$, and the function $v$ will have a unique discontinuity at $0$; then the function $u$ will be constant on $[v(0), v(0+)]$. The equation (3.39) has the form

$$\tag{3.40} x(t) = y(v(t)), \quad \frac{dy}{dt'} = D[F(y, u(t)) \ \text{on} \ \ [-T, v(0)] \cup [v(0+), T]],$$

$$\frac{dy}{dt'} = D[g(y) \ \Psi(t)] \ \text{on} \ \ [v(0), v(0+)].$$

Since the function $\Psi$ is Lipschitzian, it is absolutely continuous and has a.e. a derivative $\Psi'(t) = \psi(t)$. Using Theorem 4A.1 in [S1] and the fact that the function $v$ is continuous on $[-T, 0] \cup (0, T]$, we find an equivalent form for (3.40):

$$\frac{dx}{dt} = DF(x, t) \ \text{on} \ \ [-T, 0] \cup (0, T] ;$$

$$x(0) = y(v(0)) \ \text{and} \ x(0+) = y(v(0+)), \quad \frac{dy}{dt} = g(y) \psi(t) .$$

Notice that the equation $dy/dt = g(y) \psi(t)$ need not have the uniqueness property and the function $\psi$ may depend on the subsequence $(\Psi_n)$.

Let us return to the former case of an arbitrary sequence $(\phi_n)$ satisfying the condition (iii), moreover assuming that for every $x \in \Omega$ the ordinary differential equation
(3.41) \[ \dot{y} = g(y) \]

has a unique maximal solution such that \( y(0) = x \), and this will be denoted by \( x(t, x) \).

Denoting \( H(x, t) = F(x, u(t)) + g(x) \psi(t) \), let us describe the set \( R_{(H, v)} \):

Let \( (x, t) \in G \) be given such that \( v(t) < v(t+) \). Since the function \( u \) is constant on \([v(t), v(t+)]\), the equation \( \frac{dy}{dt'} = DH(y, t') \) will have the form \( \frac{dy}{dt'} = g(y) \psi(t') \) on \([v(t), v(t+)]\), (all solutions of the equation \( \frac{dy}{dt'} = DH(y, t') \)) are continuous, consequently we do not need the solution \( y \) on an interval \([v(t), v(t + \delta)]\).

The function \( y(s) = x(\Psi(s) - \Psi(v(t)), x) \) is a unique solution of the initial value problem

\[
\frac{dy}{ds} = g(y) \psi(s), \quad y(v(t)) = x;
\]

the function \( y \) is defined on \([v(t), v(t+)]\) if the function \( \chi(\cdot, x) \) is defined on the set

\[ \{ \tau \in R; \tau = \Psi(s) - \Psi(v(t)) \text{ for some } s \in [v(t), v(t+)] \}. \]

If \( y \) is the unique solution of (3.42) on \([v(t), v(t+)]\) then \((x, t) \) belongs to \( R_{(H, v)} \) and we denote

\[ p(x, t) = \chi(\Psi(v(t+)) - \Psi(v(t)), x) - x = \chi(\Phi(t+) - \Phi(t), x) - x. \]

Then the reduction \( F \) of the function \( H \) by \( v \) which is defined in (1.10) will have the following form:

For every \( x \in \Omega \) the continuous part of \( F(x, \cdot) \) is equal to \( F(x, \cdot) + g(x) \Phi^c(\cdot) \), and \( F(x, t+) - F(x, t) = \chi(\Phi(t+) - \Phi(t), x) - x, (x, t) \in R_{(H, v)}. \)

Let us mention that the set \( R_{(H, v)} \) can depend on the choice of the subsequence \( \Psi_{n_k} \) which converges to \( \Psi \), but the values \( F(x, t+) - F(x, t) \) do not.

Let us denote by \( R \) the set of all pairs \((x, t) \in G \) such that the function \( \chi(\cdot, x) \) is defined on the interval \([-\varepsilon(v(t+) - v(t)), \varepsilon(v(t+) - v(t))]. \)

Let us define

\[
F_0(x, t) = F(x, t) + g(x) \Phi^c(t) + \sum_{\tau \in R} \chi(\Phi(t+) - \Phi(t), x) - x.
\]

Then the sequence of functions \( F_n(x, t) = F(x, t) + g(x) \Phi_n(t) \) converges \( R \)-epathically to the function \( F_0. \)

References


Souhrn

SPojitá závislost řešení zobecněných diferenciálních rovnice na parametru

Dana Fraňková

V teorii zobecněných diferenciálních rovnice se vyskytuje zajímavý konvergenční efekt, který byl popsán J. Kurzweilem jako R-emfatická konvergence. S použitím pojmu zobecněné diferenciální rovnice se substitucí bude definována tzv. konvergence se substitucí, o niž se ukáže, že je velmi podobná R-emfatické konvergence. Posloupnost rovnice, která je konvergentní se substitucí, se dá převést na jinou posloupnost rovnice, která ke své limitní rovnici konverguje klasickým způsobem, tj. se stejnoměrnou konvergence řešení a pravých stran těchto rovnice.

Резюме

Непрерывная зависимость от параметра решений обобщенных дифференциальных уравнений

Dana Fraňková

В теории обобщенных дифференциальных уравнений появляется интересный эффект, который был описан Я. Курцвейлом как R-эмфатическая сходимость. В статье при помощи понятия обобщенного дифференциального уравнения с подстановкой определяется так называемая сходимость с подстановкой и показывается, что она очень похожа на R-эмфатическую сходимость. Последовательность уравнений, которая сходится с подстановкой, можно перевести на другую последовательность уравнений, которая сходится к своему предельному уравнению в классическом смысле, т.е. решения и правые части этих уравнений сходятся равномерно.

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