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ON DIFFERENTIAL INCLUSIONS WITH PRESCRIBED SOLUTIONS

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Summary. A new a simpler solution of the following problem is presented: given a set of absoutely continuous functions $z: J_z \to R^n$, being intervals, find the minimal multifunction F such that all functions z are solutions of the differential inclusion $\dot{x} \in F(t, x)$. (Originally the problem was solved in papers by J. Jarník and J. Kurzweil.)

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1. INTRODUCTION

The following problem was treated in [2]: Given a family $\mathscr{F}$ of absolutely continuous functions $z: J_z \to \mathbb{R}^n$, defined on intervals J_z, find a minimal multifunction F such that all those functions are Carathéodory solutions of the differential inclusion

$$\dot{x} \in F(t, x) \tag{1}$$

– the values of F should be closed, convex and F upper semicontinuous in x.

The obvious way one could use starting the construction of F is taking at every (t, x) the union

$$H(t, x) = \{\dot{z}(t): z \in \mathscr{F},\ z(t) = x,\ \dot{z}(t) \text{ exists}\}.$$

However, in the definition of Carathéodory, the existence of $\dot{z}(t)$ and the relation $\dot{z}(t) \in F(t, z(t))$ are supposed to be satisfied almost everywhere. Hence, for non denumerable $\mathscr{F}$ the union $H(t, x)$ could be, a priori, much bigger then the minimal multifunction that we are looking for. This difficulty was circumvented in [2] in the following way. A distance was introduced in $\mathscr{F}$ which became then a separable metric space – it contains a dense subset $\{z_k: k \in \mathbb{N}\}$. For every (t, x) and natural p, $H_p(t, x)$ is the closed, convex hull of all $\dot{z}_k(t)$ such that $|z_k(t) - x| \leqq 1/p$. $F(t, x)$ is the intersection of all $H_p(t, x)$.

The aim of this paper is to show that the "obvious" way of solving the problem is possible. This is done with the use of a lemma cited later which is analogous to one of the theorems of Scorza-Dragoni [5]. In the last part of the paper we propose another version of the proof from [2] – F is constructed directly, without taking the intersections.

The contents of Section 3 was included in [1] – the unpublished post-graduation thesis of the first of the authors.

2. NOTATION AND LEMMA

$\mathscr{P}(\mathbb{R}^n)$ will denote the family of all subsets of $\mathbb{R}^n$, $\mathrm{Cl}(\mathbb{R}^n)$ the family of closed subsets, $\mathrm{Conv}(\mathbb{R}^n)$ the family of compact, convex subsets.

Multifunctions are applications whose values are subsets of some space.

Let S be a multifunction defined on a metric space X with closed values in another metric space Y.

S is said to be upper semicontinuous if the set $\{x \in X: S(x) \subset U\}$ is open for every open subset U of Y.

The graph of S is the set defined by

$$\mathrm{Gr}(S) = \{(x, y) \in X \times Y: y \in S(x)\}\ .$$

It is known that if S is upper semicontinuous then $\mathrm{Gr}(S)$ is closed in $X \times Y$. If all values of S are contained in a common compact set then the inverse also holds.

Let $G \subset \mathbb{R} \times \mathbb{R}^n$ and $F: G \to \mathscr{P}(\mathbb{R}^n)$. An absolutely continuous function $x: [a, b] \to \mathbb{R}$ is a solution in the sense of Carathéodory of the differential inclusion (1) if $\dot{x}(t) \in F(t, x(t))$ almost everywhere in $[a, b]$.

We shall say that F is bounded by a locally integrable function if for some locally integrable $g: \mathbb{R} \to \mathbb{R}^+$ the condition $|y| < g(t)$ is true for all $y \in F(t, x)$.

Let $x: [a, b] \to \mathbb{R}^n$, $t \in [a, b]$. A contingent $D\,x(t)$ is the set of all limits of $(x(t_s) - x(t))/(t_s - t)$ for all sequences (t_s) converging to t. The derivative $\dot{x}(t)$ exists iff $D\,x(t)$ contains exactly one point.

The following lemma is derived from a result which was first proved in [3] and next in [4].

Let $F: \mathbb{R} \times \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ be bounded by a locally integrable function and let $F(t, \cdot): \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ be upper semicontinuous for almost all t. ($F(t, \cdot)$ is defined by $F(t, \cdot)(x) = F(t, x)$.)

Lemma. *Under the above assumptions there exists a set $T \subset \mathbb{R}$ of measure zero such that for every solution $x: [a, b] \to \mathbb{R}^n$ of (1) the condition*

$$\emptyset \neq D\,x(t) \subset F(t, x(t))$$

holds for every $t \in [a, b] \setminus T$.

3. CONSTRUCTION OF A MINIMAL MULTIFUNCTION

Let $\mathscr{F}$ be, as before, a family of absolutely continuous functions $z: J_z \to \mathbb{R}^n$, where J_z are intervals. We suppose that there exists a locally integrable function

$g: \mathbb{R} \to \mathbb{R}^+$ such that for every $z \in \mathscr{F}$ the inequality $|\dot{z}(t)| \leqq g(t)$ is true almost everywhere in J_z.

Under this assumption the following theorem is true:

Theorem. *There exists a multifunction $F: \mathbb{R} \times \mathbb{R}^n \to \text{Conv}(\mathbb{R}^n)$ such that:*

1. *every $z \in \mathscr{F}$ is a solution of (1);*
2. *$F(t, \cdot): \mathbb{R}^n \to \text{Conv}(\mathbb{R}^n)$ are upper semicontinuous for almost all $t \in \mathbb{R}$;*
3. *F is minimal i.e. for every $P: \mathbb{R} \times \mathbb{R}^n \to \text{Conv}(\mathbb{R}^n)$ such that $P(t, \cdot)$ are upper semicontinuous and all $z \in \mathscr{F}$ are solutions of $\dot{x} \in P(t, x)$ we have $F(t, x) \subset \subset P(t, x)$ for almost all $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$.*

Proof. Let us put

$$H(t, x) = \{\dot{z}(t): z \in \mathscr{F},\ z(t) = x,\ \dot{z}(t) \text{ exists}\}.$$

We define a multifunction $\tilde{H}: \mathbb{R} \times \mathbb{R}^n \to \text{Cl}(\mathbb{R}^n)$ whose graphs for fixed t are the closures in $\mathbb{R}^n \times \mathbb{R}^n$ of the graphs of $H(t, \cdot)$, i.e.

$$\text{Gr}(\tilde{H}(t, \cdot)) = \text{Cl}(\text{Gr}(H(t, \cdot))).$$

Let us consider an auxiliary multifunction $Q: \mathbb{R} \times \mathbb{R}^n \to \text{Conv}(\mathbb{R}^n)$ defined by

$$Q(t, x) = \{y \in \mathbb{R}^n: |y| \leqq g(t)\}.$$

All $z \in \mathscr{F}$ are solutions of $\dot{x} \in Q(t, x)$ and Lemma can be applied: there exists a set T of measure zero such that if $\dot{z}(t)$ exists and $t \notin T$ then $|\dot{z}(t)| \leqq g(t)$. Thus, if $t \notin T$ then all values $H(t, x)$ are contained in a ball of radius $g(t)$. This implies that $\tilde{H}(t, \cdot)$ are upper semicontinuous because the graphs $\text{Gr}(\tilde{H}(t, \cdot))$ are closed.

We put

$$F(t, x) = \text{conv}(\tilde{H}(t, x))$$

– conv stands for the closed, convex hull in $\mathbb{R}^n$. $F(t, \cdot)$ are upper semicontinuous if $t \notin T$.

It is evident that every $z \in \mathscr{F}$ is a solution of $\dot{x} \in F(t, x)$. We shall prove that F is minimal.

Let $P: \mathbb{R} \times \mathbb{R}^n \to \text{Conv}(\mathbb{R}^n)$ be upper semicontinuous with respect to $x \in \mathbb{R}^n$ and let all $z \in \mathscr{F}$ be solutions of $\dot{x} \in P(t, x)$. The multifunction

$$P'(t, x) = P(t, x) \cap Q(t, x)$$

has the same properties and Lemma can be applied to it – there is U of measure zero such that if $t \notin U$, $z \in \mathscr{F}$ and $\dot{z}(t)$ exists then $\dot{z}(t) \in P'(t, z(t))$. This implies, in view of the definition of H, that if $t \notin U$ then $H(t, x) \subset P'(t, x) \subset P(t, x)$ for all x. The graphs $\text{Gr}(P(t, \cdot))$ are closed thus $\tilde{H}(t, x) \subset P(t, x)$. The sets $P(t, x)$ are convex which finally implies that $F(t, x) \subset P(t, x)$ for $t \in R \setminus U$ and $x \in \mathbb{R}^n$.

4. ANOTHER PROOF

We shall sketch here another method of constructing the minimal multifunction from Theorem. It is based on the separability of a certain space as in [2], but F will be constructed directly, not as an intersection of a sequence of multifunctions.

$\mathscr{F}$ will be as in Theorem, but to avoid the technical difficulties we suppose that all $z \in \mathscr{F}$ are defined on $[0, 1]$.

Let $\Phi = \{(z_p, \dot{z}_p): p \in \mathbb{N}\}$ be a dense subset of $\{(z, \dot{z}): z \in \mathscr{F}\}$ – we consider it in the space $C([0, 1]) \times \mathscr{L}_1([0, 1])$, where $C([0, 1])$ is equipped with the max norm and $\mathscr{L}_1([0, 1])$ with the integral one.

We put

$$\mathrm{Gr}(H(t, \cdot)) = \mathrm{Cl}(\{z_p(t), \dot{z}_p(t)): p \in \mathbb{N}\})$$

– the closure in $\mathbb{R}^n \times \mathbb{R}^n$. The formula

$$F(t, x) = \mathrm{conv}(H(t, x))$$

provides the minimal multifunction.

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Souhrn

O DIFERENCIÁLNÍCH ŘEŠENÍCH S PŘEDEPSANÝMI ŘEŠENÍMI

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V práci je novým jednodušším způsobem řešen následující problém: Nechť je dána množina absolutně spojitých funkcí $z: J_z \to R^n$, kde J_z jsou intervaly. Najděte minimální multifunkci F tak, aby všechny funkce z byly řešeními diferenciální inkluze $\dot{x} \in F(t, x)$. (Původně byl tento problém řešen v pracích J. Jarníka a J. Kurzweila.)

Резюме

О ДИФФЕРЕНЦИАЛЬНЫХ ВКЛЮЧЕНИЯХ С ЗАДАННЫМИ РЕШЕНИЯМИ

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В работе новым, более простым способом решена следующая проблема: Пусть задано множество абсолютно непрерывных функций $z: J_z \to \boldsymbol{R}^n$, где J_z — интервалы. Определите минимальную многозначную функцию F, для которой все функции являются решениями дифференциального включения $\dot{x} \in F(t, x)$. (Эта проблема была первоначально решена в работах И. Ярника и Я. Курцвейла.)

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