Józef Myjak
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A REMARK ON SCORZA-DRAGONI THEOREM FOR DIFFERENTIAL INCLUSIONS

JÓZEF MYJAK, Trieste
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Summary. A new simple proof of the Scorza-Dragoni theorem for differential inclusions originally proved by Kurzweil and Jarník, is given.

Keywords: differential inclusion, Scorza-Dragoni theorem.

AMS Classification: 34E60.

1. INTRODUCTION

Let \( G \subset \mathbb{R} \times \mathbb{R}^d \). Let \( \mathcal{K} \) be the set of all non-empty closed convex subsets of \( \mathbb{R}^d \). Let \( S(x, r) \) denote the open ball in \( \mathbb{R}^d \) with center at \( x \) and radius \( r > 0 \). For \( \Delta \subset \mathbb{R} \) let \( \mu(\Delta) \) denote the Lebesgue measure of \( \Delta \).

Let \((Y, d)\) be a metric space. Recall that \( F: Y \rightarrow \mathcal{K} \) is called closed (or closed graph) if the set graph \( F = \{(y, z) : z \in F(y), y \in Y\} \) is closed in \( Y \times \mathbb{R}^d \). \( F \) is called upper semicontinuous (u.s.c.) at a point \( y_0 \in Y \) if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( d(y, y_0) < \delta \) implies \( F(y) \subset F(y_0) + \varepsilon S \) (\( S = S(0, 1) \)). \( F \) is called u.s.c. if it is u.s.c. at each point of \( Y \). Note that each u.s.c. multifunction with closed values is necessarily closed. The reverse is true if, in addition, the set \( F(Y) \) is relatively compact.

A multifunction \( F: [a, b] \rightarrow \mathcal{K} \) is called (Lebesgue) measurable if the set \( \{t \mid F(t) \cap \cap D \neq \emptyset \} \) is (Lebesgue) measurable for every closed subset \( D \) of \( \mathbb{R}^d \).

For a given multifunction \( F: G \rightarrow \mathcal{K} \) consider the differential inclusion

\begin{equation}
\dot{x} \in F(t, x).
\end{equation}

By a solution of (1) we mean an absolutely continuous function \( u: [a, b] \rightarrow \mathbb{R}^d \) with graph contained in \( G \) and such that \( u'(t) \in F(t, u(t)) \) for a.a. \( t \in [a, b] \).

For any function \( u: J \rightarrow \mathbb{R}^d (J \subset \mathbb{R}) \) and \( t_0 \in J \) denote by \( \text{Cont} \ u(t_0) \) the set of all \( z \in \mathbb{R}^d \) such that \( z = \lim (u(t_n) - u(t_0)) / (t_n - t_0) \) for some \( \{t_n\} \subset J, t_n \neq t_0, t_n \to t_0 \).

In [1] J. Jarník and J. Kurzweil established the following version of Scorza-Dragoni Theorem [5].

**Theorem 1.** Let \( G \) and \( \mathcal{K} \) be as above. Suppose that \( F: G \rightarrow \mathcal{K} \) is such that ...
(i) $F(t, \cdot)$ is closed for a.a. $t$ in $\text{proj}_G$;
(ii) for every $(t_0, x_0) \in G$ there exist numbers $\delta_1, \delta_2 > 0$ and an integrable function $m: [t_0 - \delta_1, t_0 + \delta_2] \to [0, +\infty)$ such that $|F(t, x)| \leq m(t)$ for every $(t, x) \in [t_0 - \delta_1, t_0 + \delta_1] \times S(x_0, \delta_2).

Then there exists a set $Q \subset R$ with $\mu(Q) = 0$ such that for every solution $u: J \to R^d$ of (1) and every $t \in J \setminus Q$ we have $0 \not\in \text{Cont} u(t) \subset F(t, u(t)).$

The original proof is based on a rather difficult approximation technique. In this note, we give a simpler and shorter proof, by combining some ideas of Opial [3] and Jarnik and Kurzweil [1, 2].

Remark 1. Theorem 1 is a slight generalization of Jarnik and Kurzweil result. In fact, in [1] $F$ is supposed to be (non-empty convex) compact valued. Moreover, in stead of condition (i) it is supposed that: (i') for every $\varepsilon > 0$ there is a measurable set $A_\varepsilon \subset R$ with $\mu(R \setminus A_\varepsilon) < \varepsilon$ such that $F$ restricted to $G \cap (A_\varepsilon \times R^d)$ is u.s.c. It is easy to see (using the projection theorem) that each $F$ satisfying (i') is Carathéodory, i.e. $F(\cdot, x)$ is (Lebesgue) measurable for each $x$, and $F(t, \cdot)$ is u.s.c. for a.a. $t$. Thus (i') implies (i), while (i) does not imply (i').

Finally let us remark that the assumption of Theorem 1 does not assure the existence of solutions of (1).

2. Proof of Theorem 1. Following [1] (owing Lindelöf property) it suffices to prove the following local version of Theorem 1.

**Theorem 2.** Let $U$ be an open subset of $R^d$. Let $I = [a, b]$. Let $F: I \times U \to \mathcal{X}$ be such that $F(t, \cdot)$ is closed for a.a. $t$ in $I$ and $|F(t, x)| \leq m(t)$ for a.a. $t \in I$ and all $x \in U$, where $m$ is integrable on $I$.

Then there is a set $I_0 \subset I$ with $\mu(I_0) = 0$ such that for every solution $u: J \to R^d$ $(J \subset I)$ of (1) and every $t \in J \setminus I_0$ we have $0 \not\in \text{Cont} u(t) \subset F(t, u(t))$.

**Proof.** By [4, Theorem 1] there exists a multifunction $\tilde{F}: I \times U \to \mathcal{X} \cup \{\emptyset\}$ such that:

(a) $\tilde{F}(t, x) \subset F(t, x)$ for every $(t, x) \in I \times U$;

(b) if $A \subset I$ is a measurable set, $u, v: D \to R^d$ are measurable functions, then $v(t) \in \tilde{F}(t, u(t))$ a.e. in $A$ implies $u(t) \in \tilde{F}(t, u(t))$ a.e. in $A$;

(c) for every $\varepsilon > 0$ there is a closed set $I_\varepsilon \subset I$ with $\mu(I \setminus I_\varepsilon) < \varepsilon$ such that $\tilde{F}$ restricted to $I_\varepsilon \times U$ is closed.

By virtue of (a) and (b) it suffices to verify the statement of Theorem 2 for $\tilde{F}$.

Let $\varepsilon > 0$. Let $I_\varepsilon$ be as in (c). By virtue of Lusin's Theorem we can assume that $m$ restricted to $I_\varepsilon$ is continuous. Thus $M = \sup \{m(t): t \in I_\varepsilon\} < +\infty$.

Denote by $\chi$ the characteristic function of the set $I \setminus I_\varepsilon$. Clearly, for a.a. $t \in I_\varepsilon$ we have

\[
\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \chi(s) m(s) \, ds = \frac{d}{dt} \int_a^t \chi(s) m(s) \, ds = 0.
\]
Let $I^*_x$ denote the set of all points of (Lebesgue) density of $I_x$ for which (2) is fulfilled. Let $t^* \in I^*_x$. Let $u: J \to \mathbb{R}^d$ be a solution of (1) (if (1) has no solution there is nothing to prove).

Claim 1. $\text{Cont } u(t^*) \neq \emptyset$.

Indeed, let $v: J \to U$ be a measurable function such that $v(s) \in F(t, u(s))$ for every $s \in J$ and

$$u(t) = u(t_0) + \int_{t_0}^t v(s) \, ds, \quad t \in J.$$  

We have

$$\frac{u(t^* + h) - u(t^*)}{h} = \frac{1}{h} \int_{t^*}^{t^* + h} \chi(s) \, v(s) \, ds + \frac{1}{h} \int_{t^*}^{t^* + h} (1 - \chi(s)) \, v(s) \, ds.$$  

From (2) and the boundedness of $m$ on $I_x$ it follows that

$$\left| \frac{u(t^* + h) - u(t^*)}{h} \right| \leq M + 1 \quad \text{for } 0 < h \leq h_0, \quad h_0 > 0.$$  

Consequently, there is a sequence $\{h_i\} \subset (0, h_0]$ with $h_i \to 0$ such that the sequence $\{(u(t^* + h_i) - u(t^*))/h_i\}$ is convergent. Thus $\text{Cont } u(t^*) \neq \emptyset$.

Claim 2. $\text{Cont } u(t^*) \subseteq F(t, u(t^*))$.

Indeed, let $z \in \text{Cont } u(t^*)$. Let $\{t^* + h_i\} \subset J$, $h_i \to 0$ be such that

$$\frac{u(t^* + h_i) - u(t^*)}{h_i} \to z.$$  

Suppose that $h_i > 0$, $i = 1, 2, \ldots$ (in the case $h_i < 0$ the arguments is similar). Set $A^*_i = [t^*, t^* + h_i] \cap I_x$. As above, we have

$$\frac{u(t^* + h_i) - u(t^*)}{h_i} = \frac{1}{h_i} \int_{t^*}^{t^* + h_i} (1 - \chi(s)) \, v(s) \, ds + \frac{1}{h_i} \int_{t^*}^{t^* + h_i} \chi(s) \, v(s) \, ds =$$

$$= \frac{\mu(A^*_i)}{h_i} \int_{A^*_i} v(s) \, ds + \frac{1}{h_i} \int_{t^*}^{t^* + h_i} \chi(s) \, v(s) \, ds.$$  

By (2) the last term in (3) tends to zero as $i \to +\infty$. Moreover, $m(A^*_i)/h_i \to 1$ as $i \to +\infty$, because $t^*$ is a point of density of $I_x$.

Since $F(\cdot, u(\cdot))$ is closed and uniformly bounded on $J \cap I_x$, it is u.s.c. on $J \cap I_x$. Thus, for given $\eta > 0$ there is $i_0$ such that $F(t, u(t)) \subseteq F(t^*, u(t^*)) + \eta S$ for every $t \in A^*_i$, $i \geq i_0$. This and the mean value theorem imply

$$\frac{1}{\mu(A^*_i)} \int_{A^*_i} v(s) \, ds \in F(t^*, u(t^*)) + \eta S, \quad i \geq i_0.$$  

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Since \( \eta \) is arbitrary, we have
\[
\lim_{t \to +\infty} \frac{1}{\mu(A_t)} \int_{A_t} v(s) \, ds \in \mathcal{F}(t^*, u(t^*)).
\]
Consequently, \( z \in \mathcal{F}(t^*, u(t^*)) \). Since \( z \) is arbitrary in \( \text{Cont} \, u(t^*) \), Claim 2 is proved.

Let \( \epsilon_n \downarrow 0 \). Set \( I^* = \bigcup I_{\epsilon_n} \). Since \( \mu(I^*) \geq \mu(I) - \epsilon_n \), we have \( \mu(I^*) = \mu(I) \). Clearly, for every solution \( u: J \to \mathbb{R}^d \) of (1) and every \( t^* \in J \cap I^*, \ 0 \not\in \text{Cont} \, u(t^*) \subseteq \mathcal{F}(t^*, u(t^*)) \). This completes the proof.

**Remark 2.** Theorem 2 fails if we drop the assumption that \( F \) is convex valued. Indeed, let \( v: [0, 1] \to \mathbb{R} \) be such that \( v(t) = 1 \) if \( t \in (1/3^k, 2/3^k) \), \( v(t) = -1 \) if \( t \in (2/3^k, 3/3^k) \), \( k = 1, 2, \ldots \) and, \( v(t) = 0 \) otherwise. Obviously the function

\[
u(t) = \int_0^t v(s) \, ds
\]

is a solution of the differential inclusion

(4)
\[
 x' \in \{-1, 1\}, \quad t \in [0, 1]
\]

and \( \text{Cont} \, u(0) = [0, 1/2] \). A slight modification of the above construction furnishes a solution of (4) such that for given \( t_0 \in [0, 1] \), \( \text{Cont} \, u(t_0) = [0, 1/2] \).

**Remark 3.** Adopting the argument of [1] one can extend immediately the above result to the case of functional differential inclusions

\[
x' \in F(t, x_t)
\]

where \( x_t(\theta) = x(t + \theta), \ \theta \in [-a, 0] \) and \( F: I \times C([-a, 0], \mathbb{R}^d) \to \mathcal{X} \).

**References**


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Souhrn

POZNÁMKA KE SCORZA-DRAGONIOVĚ VĚTĚ 
PRO DIFERENCIÁLNÍ INKLUZE 

JóZEF MYJAK

Je podán nový a jednodušší důkaz Scorza-Dragoniovy věty pro diferenciální inkluze, původně dokázané J. Kurzweilem a J. Jarníkem.

Резюме

ЗАМЕЧАНИЕ ПО ТЕОРЕМЕ СКОРЦА-ДРАГОНИ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО ВКЛЮЧЕНИЯ

JóZEF MYJAK

Дано новое и более простое доказательство теоремы Скорца-Драгони для дифференциального включения, первоначально доказанной Я. Курцвейлем и И. Ярником.

Author’s address: Facoltà Ingegneria, Università dell’Aquila, 67100 L’Aquila, Italy.