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ON APPROXIMATION BY STEP MULTIFUNCTIONS WITHOUT COMPACTNESS CONDITIONS

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Summary. We consider a pointwise approximation of semicontinuous multifunctions by nets and by sequences of step multifunctions defined on arbitrary (not necessarily compact) topological spaces. We give an upper approximation for multifunctions with values in normal spaces and a lower approximation for those with values in normed spaces.

Keywords: Upper (lower) semicontinuous multifunction, step multifunction, approximation

AMS classification: 54C60.

1. INTRODUCTION

G. Beer in [1] considered an approximation of semicontinuous multifunctions on a rectangular parallelepiped $X \subset \mathbb{R}^n$ with values in a metric space by a sequence of semicontinuous step multifunctions. In [9] we considered the same type of approximation in a more general setting. Assuming that the multifunctions are defined on compact spaces we established approximations by nets of step multifunctions. In this paper using a new type of approximation we give further results and, in particular, we improve some results of Beer.

Let us recall some definitions and facts from the theory of semicontinuous multifunctions (see [4], [5]). By a multifunction from a set X to another set Y we mean any mapping from X to the family of all subsets of Y.

Let X and Y be topological spaces and F a multifunction from X to Y. F is said to be f-upper semicontinuous (f-lower semicontinuous) if for each closed (open) set H in Y the set $F^{-1}(H)$ is closed (open) in X, where $F^{-1}(H)$ is the set of all $x \in X$ such that $F(x) \cap H \neq \emptyset$.

Assuming that Y is a quasi-uniform space we can define another notion of semicontinuity (note that every topological space is quasi-uniformizable [7]). Let $\mathcal{U}$ be a quasi-uniformity of Y. F is said to be u-upper semicontinuous (u-lower semicontinuous) at $x_0 \in X$ if for each $W \in \mathcal{U}$ there exists a neighbourhood U of x_0 such that $F(x) \subset W(F(x_0))$ ($F(x_0) \subset W^{-1}(F(x))$) for each $x \in U$, where $W(A)$ is the set of all $y \in Y$ such that $(z, y) \in W$ for some $z \in A$ and W^{-1} is the set of all $(z, y) \in Y \times Y$

such that $(y, z) \in W$. F is said to be u-upper semicontinuous (u-lower semicontinuous) if F is u-upper semicontinuous (u-lower semicontinuous) at every point of X.

It is known that 1) f-upper semicontinuity implies u-upper semicontinuity, and 2) u-lower semicontinuity implies f-lower semicontinuity. The converse implication do not hold in general. However, 1') if the values of F are compact then f-upper semicontinuity is equivalent to u-upper semicontinuity, and 2') if the values of F are totally bounded then f-lower semicontinuity is equivalent to u-lower semicontinuity.

Now, let $\{F_t, t \in T\}$ be a net of multifunctions from X to Y, and let $\mathscr{U}$ be a quasi-uniformity of Y. We say (see [9]) that the net $\{F_t, t \in T\}$ is an upper approximation (lower approximation) of a multifunction F from X to Y if the following two conditions hold:

i) for each $x \in X$ and every $t, t' \in T$ with $t \leqq t'$ we have $F(x) \subset F_{t'}(x) \subset F_t(x)$ ($F_t(x) \subset F_{t'}(x) \subset F(x)$),

ii) for each $W \in \mathscr{U}$ and each $x \in X$ there exists $t_0 \in T$ such that for each $t \geqq t_0$ we have $F_t(x) \subset W(F(x))$ ($F(x) \subset W^{-1}(F_t(x))$).

Note that if the space Y is uniformizable then W^{-1} can be replaced by W. However, in general, the use of W^{-1} instead of W is more useful.

2. APPROXIMATION BY NETS

In this section we prove two theorems on approximation by nets of step multifunctions, i.e. by multifunctions which assume only a finite number of values. The constructions of these approximations are different from those in [9].

Theorem 1. *Let X be a topological space and Y a normal space, and let $\mathscr{U}(Y)$ be a uniformity of Y. Suppose that F is a u-upper semicontinuous multifunction from X to Y. Then F has an upper approximation by a net of f-upper semicontinuous closed-valued step multifunctions.*

Proof. Let P be the family of all finite subsets of X and let $\mathscr{U}(X)$ be a quasi-uniformity of X. Define T to be the set $P \times \mathscr{U}(X)$. T is a directed set under the relation

$$(A, V) \leqq (A', V') \quad \text{if and only if} \quad A \subset A' \quad \text{an}\beta \quad V' \subset V .$$

For each $(A, V) \in T$ and each $s \in A$ we define

$$\Theta_{s,V}(x) = \mathrm{cl}\Big(\bigcup_{z \in V(s)} F(z)\Big) \quad \text{if} \quad x \in \operatorname{int} V(s) ,$$
$$= Y \quad \text{otherwise in} \quad X .$$

Each multifunction $\Theta_{s,V}$ is f-upper semicontinuous and closed-valued. Now, for each $t = (A, V) \in T$ we define

$$F_t(x) = \bigcap_{s \in A} \Theta_{s,V}(x) \quad \text{for} \quad x \in X .$$

Since the space Y is normal, the multifunctions F_t are f-upper semicontinuous ([3], Theorem 1, p. 179). Moreover, they are closed-valued step multifunctions.

For each $x \in X$ and $s \in A$ we have $F(x) \subset \Theta_{s,V}(x)$. This implies that $F(x) \subset F_t(x)$ for every $t \in T$ and every $x \in X$. If $t \leqq t'$ where $t = (A, V)$ and $t' = (A', V')$ then

$$\bigcup_{z \in V'(s)} F(z) \subset \bigcup_{z \in V(s)} F(z) ,$$

which implies that $F_{t'}(x) \subset F_t(x)$.

It remains to prove the convergence condition ii). Take an arbitrary $x_0 \in X$ and an arbitrary $W \in \mathscr{U}(Y)$. Since F is u-upper semicontinuous there exists $V_0 \in \mathscr{U}(X)$ such that

$$\mathrm{cl}\Big(\bigcup_{z \in V_0(x_0)} F(z)\Big) \subset W(F(x_0))$$

Take $t_0 = (A_0, V_0)$ where $A_0 = \{x_0\}$. Then

$$\Theta_{x_0,V_0}(x_0) = \mathrm{cl}\Big(\bigcup_{z \in V_0(x_0)} F(z)\Big) \subset W(F(x_0)) .$$

Therefore for each $t \geqq t_0$ we obtain $F_t(x_0) \subset W(F(x_0))$. This complets the proof.

For the proof of Theorem 2 below which concerns lower approximation we need the following results:

Proposition 1 ([6]). *Let Y be a normed space and B a bounded convex subset of Y with a nonempty interior. Then for an arbitrary neighbourhood W of O in Y there exist a subset Z of B and a neighbourhood U of O in Y such that*

$$Z + U \subset B \subset Z + W .$$

Proposition 2 (the law of cancellation, see [8], [10]). *Let Y be a topological vector space, A an arbitrary subset of Y, B a bounded subset of Y and C a nonempty convex and closed subset of Y. Then*

$$A + B \subset \mathrm{cl}(C + B) \quad \text{implies} \quad A \subset C .$$

Remark that whenever Y is a topological vector space we take as $\mathscr{U}(Y)$ the natural uniformity of Y with the base $\mathscr{B}$, where $W \in \mathscr{B}$ if and only if W is the set of all $(y, z) \in$ $\in Y \times Y$ with $y - z \in V$ where V is a neighbourhood of O in Y. Accordingly, we will write $Z + V$ rather than $W(Z)$.

Theorem 2. *Let X be a topological space, Y a normed space and F a u-lower semicontinuous multifunction from X to Y wuch that the values $F(x)$ are closed,* *bounded, convex and have nonempty interiors. Then F has a lower approximation by a net of f-lower semicontinuous closed-valued step multifunctions.*

Proof. Let T be the directed set defined in the proof of Theorem 1. For each $(A, V) \in T$ and $s \in A$ we define

$$\Theta_{s,V}(x) = \bigcap_{z \in V(s)} F(z) \quad \text{if} \quad x \in \operatorname{int} V(s)\,,$$

$$= \emptyset \quad \text{otherwise in} \quad X\,.$$

The multifunctions $\Theta_{s,V}$ are f-lower semicontinuous and closed-valued. Next, for each $t = (A, V) \in T$ we define

$$F_t(x) = \bigcup_{s \in A} \Theta_{s,V}(x) \quad \text{for} \quad x \in X\,.$$

The multifunctions F_t are f-lower semicontinuous closed-valued step multifunctions. Moreover, $F_t(x) \subset F(x)$ for each $x \in X$ and $t \in T$ since $\Theta_{s,V}(x) \subset F(x)$, and $F_t(x) \subset \subset F_{t'}(x)$ for each $x \in X$ and $t \leqq t'$ since $\Theta_{s,V}(x) \subset \Theta_{s,V'}(x)$ where $t = (A, V)$ and $t' = (A', V')$. It remains to prove the convergence.

Let $x_0 \in X$ and let W be an arbitrary neighbourhood of O in Y. By Proposition 1 we get a set $Z \subset F(x_0)$ and a bounded neighbourhood U of O in Y such that

$$Z + U \subset F(x_0) \subset Z + W\,.$$

The u-lower semicontinuity of F implies that there exists a neighbourhood $V_0 \in \mathscr{U}(X)$ such that

$$F(x_0) \subset F(z) + U \quad \text{for each} \quad z \in V_0(x_0)\,.$$

Then for each $z \in V_0(x_0)$ we have

$$Z + U \subset F(x_0) \subset F(z) + U\,.$$

The law of cancellation (Proposition 2) implies that $Z \subset F(z)$ for each $z \in V_0(x_0)$ and thus

$$Z \subset \bigcap_{z \in V_0(x_0)} F(z)\,.$$

This yields

$$F(x_0) \subset Z + W \subset \bigcap_{z \in V_0(x_0)} F(z) + W\,,$$

which means that

$$F(x_0) \subset \Theta_{x_0,V_0}(x_0) + W\,.$$

If we take $A_0 = \{x_0\}$ and $t_0 = (A_0, V_0)$ then $F(x_0) \subset F_{t_0}(x_0) + W$ which complets the proof.

3. APPROXIMATION BY SEQUENCES

We are able to obtain sequential approximations provided the space X is a totally bounded metric space (cf [2]). The approximated sequences are defined inductively.

In case of the upper approximation we proceed as follows. For $n = 1$ we find a finite set $A_1 \subset X$ which is 1-dense in X, and define

$$F_1(x) = \bigcap_{s \in A_1} \Theta_{s,1}(x) \quad \text{for} \quad x \in X\,,$$

where

$$\begin{aligned} \Theta_{s,1}(x) &= \mathrm{cl}(F(B(s, 1))) \quad \text{if} \quad x \in B(s, 1)\,, \\ &= Y \quad \text{otherwise in} \quad X\,, \end{aligned}$$

and $B(s, 1)$ is the open ball in X with center s and radius 1. If multifunctions $F_1, \ldots$ $\ldots, F_{n-1}$ are defined then we take a finite subset A_n of X such that $A_{n-1} \subset A_n$ and A_n is $1/n$-dense in X, and put

$$F_n(x) = \bigcap_{s \in A_n} \Theta_{s,n}(x) \quad \text{for} \quad x \in X$$

where

$$\begin{aligned} \Theta_{s,n}(x) &= \mathrm{cl}(F(B(s, 1/n))) \quad \text{if} \quad x \in B(s, 1/n)\,, \\ &= Y \quad \text{otherwise in} \quad X \end{aligned}$$

and $B(s, 1/n)$ is the open ball in X with center s and radius $1/n$.

In case of the lower approximation the construction of an approximating sequence proceeds as follows

$$F_n(x) = \bigcup_{s \in A_n} \Theta_{s,n}(x) \quad \text{for} \quad x \in X$$

where

$$\begin{aligned} \Theta_{s,n}(x) &= \bigcap_{z \in B(s,1/n)} F(z) \quad \text{if} \quad x \in B(s, 1/n)\,, \\ &= \emptyset \quad \text{otherwise in} \quad X \end{aligned}$$

and $A_1, A_2, \ldots$ are defined inductively as in the case of the upper approximation.

The corresponding theorems for sequential approximations are:

Theorem 3. *Let X be a totally bounded metric space, Y a normal space and F a u-upper semicontinuous multifunction from X to Y (with respect to a uniformity $\mathcal{U}$ of Y). Then F has an upper approximation by a sequence of closed-valued f-upper semicontinuous step multifunctions.*

Theorem 4. *Let X be a totally bounded metric space, Y a normed space and let F be a u-lower semicontinuous multifunction from X to Y (with respect to the natural uniformity of Y) whose values are closed, bounded and convex subsets with non-*

empty interiors. Then F has a lower approximation by a sequence of f-lower semicontinuous closed-valued step multifunctions.

The proofs of Theorems 3 and 4 except for the convergence parts run similarly the proofs of Theorems 1 and 2, respectively. The proofs of the convergence may proceed as follows:

The case of upper approximation. Given an arbitrary W and $x_0 \in X$ we use the u-upper semicontinuity of F to find a neighbourhood V of x_0 in X such that $\mathrm{cl}(F(V)) \subset \subset W(F(x_0))$. Then we take n_0 such that $x_0 \in B(s_0, 1/n_0)$ and $B(s_0, 1/n_0) \subset V$ for some $s_0 \in A_{n_0}$. We get

$$F_{n_0}(x_0) \subset \Theta_{s_0,n_0}(x_0) = \mathrm{cl}(F(B(s_0, 1/n_0))) \subset W(F(x_0))$$

which completes the proof.

The case of lower approximation. Let $x_0 \in X$ and let W be an arbitrary neighbourhood of O in Y. As in the proof of Theorem 2 we find a neighbourhood V of x_0 in X such that

$$F(x_0) \subset \bigcap_{z \in V} F(z) + W.$$

Now we take n_0 such that $x_0 \in B(s_0, 1/n_0)$ and $B(s_0, 1/n_0) \subset V$ for some $s_0 \in A_{n_0}$. Then

$$F(x_0) \subset \bigcap_{z \in B(s_0, 1/n_0)} F(z) + W \subset F_{n_0}(x_0) + W$$

which completes the proof.

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Souhrn

APROXIMACE JEDNODUCHÝMI MULTIFUNKCEMI BEZ PODMÍNEK KOMPAKTNOSTI

ANDRZEJ SPAKOWSKI

V článku se studuje aproximace polospojitých multifunkcí sítěmi a posloupnostmi polospojitých jednoduchých multifunkcí definovaných na libovolných (nikoliv nutně kompaktních) topologických prostorech. Je odvozena horní aproximace pro multifunkce s hodnotami v normálních prostorech a dolní aproximace pro multifunkce v normovaných prostorech.

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