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ON COMPLETIONS OF LINEARLY ORDERED GROUPS

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Summary. Each lattice ordered group $G$ can be associated with a class $C(G)$ of lattice ordered groups which are in a certain sense generated by $G$ (for a thorough definition cf. below). In this note we investigate the relations between $C(G)$ and the completion of $G$, where $G$ is a linearly ordered group.

Keywords: linearly ordered group, closed 1-subgroup, completion of a linearly ordered group

AMS Subject Classification: 06F15.

1. INTRODUCTION

For a lattice ordered group $G$ we denote by $m(G)$ the completion of $G$ (in the sense of [4], Chap. V, § 10; in [4], the notation $G_D$ was used). This notion was studied in [3] and [6] for the abelian case, and in [2] for the non-abelian case.

Clearly we have $m(m(G)) = m(G)$. If $m(G) = G$, then $G$ will be said to be $m$-complete. In the case when $G$ is archimedean, $m(G)$ coincides with the Dedekind completion of $d(G)$ of $G$ (cf., e.g., [1], Chap. XIII, § 13).

An 1-subgroup $G_1$ of $G$ is said to be closed if, whenever $\{x_i\}_{i \in I} \subseteq G_1$, $x \in G$ and the relation $x = \sup \{x_i\}_{i \in I}$ is valid in $G$, then $x \in G_1$. In such a case the corresponding dual condition also holds.

If $G$ is an 1-subgroup of a lattice ordered group $H$ such that for each closed 1-subgroup $H_1$ of $H$ with $G \subseteq H_1$ the relation $H_1 = H$ is valid, then we say that $H$ is a $c$-completion of $G$, or that $G$ $c$-generates $H$.

We denote by $C(G)$ the class of all lattice ordered groups $H$ such that

(i) $H$ is $m$-complete;
(ii) $G$ $c$-generates $H$.

In general, even in the case when $G$ is archimedean, $C(G)$ can contain infinitely many (in fact, a proper class of) mutually non-isomorphic lattice ordered groups (cf. [8], [9]). More thoroughly: there exists an archimedean lattice ordered group $G$ such that for each cardinal $\alpha$ there is $H \in C(G)$ with $\text{card } H \geq \alpha$. Hence, in a certain sense, $C(G)$ can be "extremely large".

A similar situation occurs in the theory of Boolean algebras [5] and of vector lattices [7].
It is obvious that \( m(G) \in C(G) \) for each lattice ordered group \( G \). We shall show that if \( G \) is a linearly ordered group, then the class \( C(G) \) is “extremally small”, namely, the following result will be proved:

\((\ast)\) Let \( G \) be a linearly ordered group. Then each element of \( C(G) \) is isomorphic to \( m(G) \).

This generalizes a result from [9] (Proposition 3.4) concerning archimedean linearly ordered groups.

2. PROOF OF \((\ast)\)

Let \( G \) be a linearly ordered group. If \( G = \{0\} \), then \((\ast)\) obviously holds. In what follows, \( G \) denotes a nonzero linearly ordered group and \( H \) a fixed element of \( C(G) \).

Let \( T_1 \) be the system of all elements \( h \in H \) such that \( h = \bigvee_{i \in I} x_i \) for some subset \( \{x_i\}_{i \in I} \) of \( G \). Next, let \( T_2 \) have the dual meaning and \( T = T_1 \cup T_2 \).

The following assertion is obvious.

2.1. Lemma. Let \( \{h_j\}_{j \in J} \subseteq T_1 \), \( \{h'_j\}_{j \in K} \subseteq T_2 \), \( h, h' \in H \), \( h = \bigvee_{j \in J} h_j \) and \( h' = \bigwedge_{k \in K} h'_k \). Then \( h \in T_1 \) and \( h' \in T_2 \). Further, \( G \subseteq T_1 \cap T_2 \).

2.2. Lemma. Let \( \emptyset \neq \{h_j\}_{j \in J} \subseteq T_1 \), \( h \in H \), \( h = \bigwedge_{j \in J} h_j \). Then \( h \in T \).

Proof. If there exists \( j \in J \) such that \( h_j = h \), then \( h \in T_1 \subseteq T \). Suppose that \( h_j > h \) for each \( j \in J \).

Let \( j \in J \). There are elements \( g_{ij} \in G \) \((i \in I(j))\) such that \( h_j = \bigvee_{i \in I(j)} g_{ij} \). If \( g_{ij} \leq h \) for each \( i \in I(j) \), then \( h_j \leq h \), which is a contradiction. Hence there is \( f(j) \in I(j) \) such that \( h < g_{f(j), j} \leq h_j \). Therefore \( h = \bigwedge_{j \in J} g_{f(j), j} \) and thus \( h \in T_2 \subseteq T \).

2.3. Lemma. Let \( \emptyset \neq \{h_j\}_{j \in J} \subseteq T \), \( h \in H \), \( h = \bigwedge_{j \in J} h_j \). Then \( h \in T \).

Proof. Denote \( J_1 = \{j \in J : h_j \in T_1 \} \), \( J_2 = \{j \in J : h_j \in T_2 \} \). If \( J_1 = \emptyset \), then in view of 2.1 we have \( h \in T_2 \). If \( J_2 = \emptyset \), then 2.2 yields that \( h \in T \).

Suppose that \( J_1 \neq \emptyset \neq J_2 \). For each \( j \in J_2 \) there is \( \{g_{ij}\}_{i \in I(j)} \subseteq G \) such that \( h_j = \bigwedge_{i \in I(j)} g_{ij} \). Then one of the following conditions is valid:

(i) \( \bigwedge_{j \in J_2, i \in I(j)} g_{ij} = h \);

(ii) there exists \( c \in H \) with \( h < c \) such that \( g_{ij} > c \) for each \( j \in J_2 \) and each \( i \in I(j) \).

Assume that (i) holds. Then \( h \in T_2 \). Next, suppose that (ii) is valid. Then the set \( J_3 = \{j \in J_1 : h_j < c \} \) is nonempty and \( h = \bigwedge_{j \in J_3} h_j \). Thus 2.2 yields that \( h \in T \).

Analogously we can verify that the assertion dual to 2.3 also holds. Hence we have

2.4. Corollary. \( T \) is a closed sublattice of \( H \).

2.5. Lemma. \( T \) is a subgroup of \( H \).
Proof. If \( k \in \{1, 2\} \) and \( x, y \in T_k \), then clearly \( x + y \in T_k \). Let \( x \in T_1 \) and \( y \in T_2 \). Hence there are \( \{x_i\}_{i \in I} \subseteq G \) and \( \{y_j\}_{j \in J} \subseteq G \) such that \( x = \bigvee_{i \in I} x_i \) and \( y = \bigwedge_{j \in J} y_j \). Then
\[
    x + y = (\bigvee_{i \in I} x_i) + (\bigwedge_{j \in J} y_j) = \bigwedge_{j \in J} \bigvee_{i \in I} (x_i + y_j).
\]
Thus in view of 2.2 we infer that \( x + y \) belongs to \( T \). Next, if \( x \in T_1 \), then \(-x \in T_2 \); analogously from \( x \in T_2 \) we obtain that \(-x \in T_1 \).

2.6. Lemma. \( T = H \).

Proof. This follows from \( H \in C(G) \) and from Lemmas 2.1, 2.4 and 2.5.

2.7. Lemma. \( T_1 = T_2 \).

Proof. It suffices to verify that \( T_1 \subseteq T_2 \). By way of contradiction, assume that there is \( h \in T_1 \) such that \( h \) does not belong to \( T_2 \). Hence \( h \notin G \) and there is an element \( c \in H \) with \( h < c \) such that no element \( g' \) of \( G \) satisfies the relation \( h < g' < c \).

Suppose that \( h' \in H \) and \( h < h' < c \). Then neither \( h' \in T_1 \) nor \( h' \in T_2 \) can be valid, which contradicts 2.6. Therefore the interval \([h, c] \) of \( H \) is a prime interval.

By applying the translation \( \psi(t) = t + (-c + h) \) (where \( t \) runs over \( H \)) we obtain that \([\psi(h), \psi(c)] \) is a prime interval in \( H \) as well; clearly \( \psi(c) = h \). This shows that \( h \) does not belong to \( T_1 \), which is a contradiction.

2.8. Lemma. Let \( h \in H \). Then there are \( X, Y \subseteq G \) such that \( \sup X = h = \inf Y \) holds in \( H \).

Proof. This is a consequence of 2.6 and 2.8.

Now let us investigate the relations between the lattice ordered groups \( m(G) \) and \( H \).

Let \( t \in m(G) \). Put \( X_t = \{g \in G : g \leq t\} \) and \( Y_t = \{g \in G : g \geq t\} \). Then in view of 1.3 in [2], the relations

- (a) \( \bigwedge \{y - x : x \in X_t, y \in Y_t\} = 0 \),
- (b) \( \bigwedge \{-x + y : x \in X_t, y \in Y_t\} = 0 \)

are valid in \( G \).

Let \( h_0 \in H \), \( h_0 \leq y - x \) for each \( x \in X_t \) and each \( y \in Y_t \). If \( h_0 > 0 \), then in view of 2.8 there is \( x_0 \in G \) with \( 0 < x_0 \leq h_0 \). Hence \( x_0 \leq y - x \) for each \( x \in X_t \) and each \( y \in Y_t \), which contradicts (a). Thus the condition

- (a₁) \( \bigwedge \{y - x : x \in X_t, y \in Y_t\} = 0 \)

is valid in \( H \).

Similarly, the condition

- (b₁) \( \bigwedge \{-x + y : x \in X_t, y \in Y_t\} = 0 \)

holds in \( H \).
Next, by virtue of the conditions \((a_1), (b_1)\) and in view of the fact that \(H\) is \(m\)-complete there is \(h_1 \in H\) such that the relations
\[
\sup X_1 = h_1 = \inf Y_1
\]
hold in \(H\). Put \(\varphi(t) = h_1\).

It is easy to verify that for \(t_1, t_2 \in H\) the equivalence
\[
t_1 \leq t_2 \iff \varphi(t_1) \leq \varphi(t_2)
\]
is valid.

Let \(h \in H\). Next, let \(X\) and \(Y\) be as in 2.8. Again, because \(m(H) = H\) and in view of 1.3 in [2] the conditions \((a_1)\) and \((b_1)\) hold for \(X\) and \(Y\) in \(H\). This yields that the conditions \((a)\) and \((b)\) are valid in \(G\) for \(X\) and \(Y\). Thus there is \(t_1 \in m(G)\) such that
\[
\sup X = t_1 = \inf Y
\]
is valid in \(m(G)\). Then we clearly have \(\varphi(t_1) = h\). Hence \(\varphi\) is a surjection.

Let \(t_2, t_3 \in m(G)\). There are \(X_2, X_3 \subseteq G\) such that the relations
\[
t_2 = \sup X_2 \quad \text{and} \quad t_3 = \sup X_3
\]
hold in \(m(G)\). This yields that
\[
\varphi(t_2) = \sup X_2 \quad \text{and} \quad \varphi(t_3) = \sup X_3
\]
are valid in \(H\). Next, we obtain that the relation
\[
t_2 + t_3 = \sup \{x_2 + x_3 : x_2 \in X_2 \text{ and } x_3 \in X_3\}
\]
holds in \(m(G)\) and that
\[
\varphi(t_2) + \varphi(t_3) = \sup \{x_2 + x_3 : x_2 \in X_2 \text{ and } x_3 \in X_3\}
\]
is valid in \(H\). Thus
\[
\varphi(t_2 + t_3) = \varphi(t_2) + \varphi(t_3).
\]
Clearly \(\varphi(g) = g\) for each \(g \in G\).

Summarizing, we have the following result (which implies that \((\ast)\) holds):

\section*{2.9. Theorem.} Let \(G\) be a linearly ordered group and let \(H \in C(G)\). Then there is an isomorphism \(\varphi\) of \(m(G)\) onto \(H\) such that \(\varphi(g) = g\) for each \(g \in G\).

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Súhrn

O ZÚPLNENIACH LIEÁRNE USPORIADANÝCH GRÚP

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Každej zvážove usporiadanej grupe G prísluša trieda C(G) zvážove usporiadanych grúp, ktoré sú v istom zmysle vytvorené grupou G. V tejto poznámke sa vyšetrujú vztahy medzi C(G) a zúplnenim G v prípade, že G je lineárne usporiadaná grupa.

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