Časopis pro pěstování matematiky

Tomáš Roubíček

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Časopis pro pěstování matematiky, Vol. 115 (1990), No. 4, 338--342

Persistent URL: http://dml.cz/dmlcz/118410

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A GENERALIZATION OF THE LIONS-TEMAM COMPACT IMBEDDING THEOREM

TOMÁŠ ROUBÍČEK, Praha

(Received August 29, 1988)

Summary. The well-known theorem by J. L. Lions and R. Temam concerning the compact imbedding of the space $\{v \in L^p(0, T; B_0);\ \mathrm{d}v/\mathrm{d}t \in L^q(0, T; B_1)\}$ into $L^p(0, T; B)$ is generalized to the case when B_0 is a reflexive Banach space imbedded compactly into a normed linear space B that is continuously imbedded into a Hausdorff locally convex space B_1, and $1 < p < +\infty$, $1 \leqq q \leqq +\infty$. Applications of such generalization to numerical analysis are outlined.

Keywords: compact imbedding, evolution equations, locally convex spaces.

AMS Subject Classification: Primary 46A50, Secondary 35K65, 65Mxx.

In [1; Chap. 1, Thm. 5.1] and [2; Chap. III, Thm. 2.1] J. L. Lions and R. Temam posed the broadly applicable theorem concerning the compact imbedding of the space

$$W^{p,q}(0, T; B_0, B_1) = \left\{v \in L^p(0, T; B_0)\ ;\ \frac{\mathrm{d}v}{\mathrm{d}t} \in L^q(0, T; B_1)\right\} \tag{1}$$

into the space $L^p(0, T; B)$, where $B_0 \subset B \subset B_1$ are three Banach spaces, B_0, B_1 are reflexive, the imbedding $B_0 \subset B$ is compact and $B \subset B_1$ is continuous, $1 < p < +\infty$, $1 < q < +\infty$, and $T > 0$. This theorem is very powerful since B_1 can be chosen arbitrarily large. The aim of this short note is to show that, in fact, it is sufficient to take for B_1 even an arbitrary locally convex space with the only condition that its topology is a Hausdorff one. Besides, q may be equal to 1 or $+\infty$ and B need not be complete. At the end of this note some applications of such generalization will be briefly outlined.

Let B_1 be a locally convex pace, $\{|\cdot|_\iota\}_{\iota \in I}$ being a collection of seminorms generating its topology (I is an index set). Let the seminorm $|\cdot|_{q\iota}$ be defined by

$$|v|_{q\iota} = \begin{cases} \left(\int_0^T |v(t)|_\iota^q \,\mathrm{d}t\right)^{1/q} & \text{if } 1 \leqq q < +\infty \quad \text{and} \\ \operatorname*{ess\,sup}_{0 \leqq t \leqq T} |v(t)|_\iota & \text{if } q = +\infty\,. \end{cases}$$

Put $L^q(0, T; B_1) = \{v\colon [0, T] \to B_1;\ v \text{ is Bochner integrable, } |v|_{q\iota} < +\infty\ \forall \iota \in I\}$. By endowing $L^q(0, T; B_1)$ with a collection of the seminorms $\{|\cdot|_{q\iota}\}_{\iota \in I}$, we obviously get a locally convex space. As usual, we will understand a linear operator to be compact if it maps bounded subsets into precompact ones.

Theorem. *Let B_0 be a normed linear space imbedded compactly into another normed linear space B which is continuously imbedded into a Hausdorff locally convex space B_1, and $1 \leqq p < +\infty$. If $v, v_i \in L^p(0, T; B_0)$, $i \in N$, the sequence $\{v_i\}_{i \in N}$ converges weakly to v in $L^p(0, T; B_0)$, and $\{dv_i/dt\}_{i \in N}$ is bounded in $L^1(0, T; B_1)$, then $\{v_i\}_{i \in N}$ converges to v strongly in $L^p(0, T; B)$.*

Proof. First we will prove that $\forall \eta > 0\ \exists J_\eta \in \mathscr{F}(I)\ \exists c_\eta \in \mathbf{R}\ \forall u \in B_0$:

$$\|u\|_B^p \leqq \eta \|u\|_{B_0}^p + c_\eta \sum_{\iota \in J_\eta} |u|_\iota^p \,, \tag{2}$$

where $\mathscr{F}(I)$ is the set of all finite subsets of I. Supposing the contrary, we get $\eta > 0$ such that $\forall J \in \mathscr{F}(I)\ \forall c \in \mathbf{R}\ \exists u_{Jc} \in B_0$: $\|u_{Jc}\|_B^p \geqq \eta \|u_{Jc}\|_{B_0}^p + c \sum_{\iota \in J} |u_{Jc}|_\iota^p$. Putting $w_{Jc} = u_{Jc}/\|u_{Jc}\|_{B_0}$, we get:

$$\|w_{Jc}\|_B^p \geqq \eta + c \sum_{\iota \in J} |w_{Jc}|_\iota^p \,, \tag{3}$$

and also $\|w_{Jc}\|_B \leqq C$, where $C = \sup_{u \neq 0} \|u\|_B / \|u\|_{B_0}$ represents the norm of the imbedding operator $B_0 \to B$. Hence $\sum_{\iota \in J} |w_{Jc}|_\iota^p \leqq C^p/c$, and thus also $|w_{Jc}|_\iota \leqq$ $\leqq C c^{-1/p}$ whenever $\iota \in J$. Thus $\lim_{c \to +\infty, J \in \mathscr{F}(I)} |w_{Jc}|_\iota = 0$ for every $\iota \in I$. Note that $\mathscr{F}(I)$ and $\mathbf{R}$ are directed by the relations $\subset$ and $\leqq$, respectively, and thus we can speak actually about the net $\{w_{Jc}\}_{J \in \mathscr{F}(I), c \in \mathbf{R}}$ and about its possible limit.

This net forms a precompact subset of B because it is bounded in B_0 which is compactly imbedded into B. Hence there is its subnet (denote it by same indices, for simplicity) such that $w_{Jc} \to w$ strongly in $\bar{B}$, where $\bar{B}$ denotes the completion of B (if B is a Banach space, then, of course, $\bar{B} = B$). As the imbedding $B \subset B_1$ is continuous, each of the seminorms $|\cdot|_\iota$ is uniformly continuous on B, and we may extend it continuously on $\bar{B}$, denoting the extension again by $|\cdot|_\iota$, for simplicity. Therefore we have $|w_{Jc} - w|_\iota \to 0$ for every $\iota \in I$. Clearly, we can write $|w|_\iota \leqq$ $\leqq |w_{Jc}|_\iota + |w_{Jc} - w|_\iota$. Passing to the limit, we get $|w|_\iota = 0$ for every $\iota \in I$. Thus $w = 0$ because the topology of B_1 has been supposed to be Hausdorff. In other words, $w_{Jc} \to 0$ strongly in B, which contradicts (3), thus proving (2).

Without loss of generality we may take $v = 0$. Let $\varepsilon > 0$. As the sequence $\{v_i\}_{i \in N}$ is bounded in $L^p(0, T; B_0)$, we can take $\eta = \varepsilon/(2 \,.\, \sup_{i \in N} \|v_i\|_{L^p(0,T;B_0)}^p)$. Integrating (2) over $[0, T]$ we get

$$\|v_i\|_{L^p(0,T;B)}^p \leqq \frac{\varepsilon}{2} + c \sum_{\iota \in J} |v_i|_{p\iota}^p$$

with some $c \in \mathbf{R}$ and $J \in \mathscr{F}(I)$ depending on ε. The proof will be completed if we show that

$$\lim_{i \to \infty} |v_i|_{p\iota}^p = 0 \quad \text{for all} \quad \iota \in I \,. \tag{4}$$

Clearly, $|v_i|_{p\iota}^p = \int_0^{T/2} |v_i(t)|_\iota^p \,dt + \int_{T/2}^T |v_i(t)|_\iota^p \,dt$ and we may investigate only the first term, while the second can be treated analogously. For every $s > 0$ such that $s \leqq T/2$ and every $t \in [0, T/2]$ we may write $v_i(t) = a_i(t) + b_i(t)$, where

$$a_i(t) = \frac{1}{s} \int_0^s v_i(t + \tau) \,d\tau \quad \text{and} \quad b_i(t) = \int_0^s \left(\frac{\tau}{s} - 1 \right) \frac{d}{dt} v_i(t + \tau) \,d\tau \,.$$

Hence

$$\int_0^{T/2} |v_i(t)|_\iota^p \, \mathrm{d}t = 2^{p-1} \int_0^{T/2} |a_i(t)|_\iota^p \, \mathrm{d}t + 2^{p-1} \int_0^{T/2} |b_i(t)|_\iota^p \, \mathrm{d}t = I_1 + I_2 .$$

We can estimate

$$I_2 \leqq 2^{p-1} \int_0^{T/2} \left(\int_0^s \left(1 - \frac{\tau}{s}\right) \left| \frac{\mathrm{d}}{\mathrm{d}t} v_i(t + \tau) \right|_\iota \mathrm{d}\tau \right)^p \mathrm{d}t =$$

$$= 2^{p-1} \left\| \left| \frac{\mathrm{d}}{\mathrm{d}t} v_i \right|_\iota * \psi_s \right\|_{L^p(0,T/2)}^p ,$$

where „*" denotes the convolution, i.e. $[f * g](t) = \int f(\tau) g(t - \tau) \, \mathrm{d}\tau$, and $\psi_s: \boldsymbol{R} \to \to \boldsymbol{R}$ is defined by

$$\psi_s(t) = \begin{cases} t/s + 1 & \text{for} \quad -s \leqq t \leqq 0 , \\ 0 & \text{elsewhere} . \end{cases}$$

The following estimates are well known: $\|f * g\|_{L^1(\boldsymbol{R})} \leqq \|f\|_{L^1(\boldsymbol{R})} \|g\|_{L^1(\boldsymbol{R})}$ and $\|f * g\|_{L^\infty(\boldsymbol{R})} \leqq \|f\|_{L^1(\boldsymbol{R})} \|g\|_{L^\infty(\boldsymbol{R})}$. As $g \mapsto f * g$ is a linear operator on $L^1(\boldsymbol{R})$, we can obtain by interpolation (using the classical Riesz-Thorin convexity theorem) the estimate

$$\|f * g\|_{L^p(\boldsymbol{R})} \leqq \|f\|_{L^1(\boldsymbol{R})} \|g\|_{L^p(\boldsymbol{R})} .$$

It yields the estimate

$$\left\| \left| \frac{\mathrm{d}}{\mathrm{d}t} v_i \right|_\iota * \psi_s \right\|_{L^p(0,T/2)} \leqq \left\| \left| \frac{\mathrm{d}}{\mathrm{d}t} v_i \right|_\iota \right\|_{L^1(0,T/2+s)} \|\psi_s\|_{L^p(\boldsymbol{R})} .$$

As $\|\psi_s\|_{L^p(\boldsymbol{R})} \leqq s^{1/p}$, we get $I_2 \leqq 2^{p-1} s |\mathrm{d}v_i/\mathrm{d}t|_{1\iota}^p$, and we see that $I_2 = \mathcal{O}(s)$ for $s \to 0$ because, by the assumptions, $\{\mathrm{d}v_i/\mathrm{d}t\}_{i \in N}$ is bounded in $L^1(0, T; B_1)$, hence particularly in the seminorm $|\cdot|_{1\iota}$. Thus the term I_2 can be made arbitrarily small when taking s small enough.

Now, let us take $s > 0$ fixed and investigate the term I_1. Since $v_i \to 0$ weakly in $L^p(0, T, B_0)$, we can see that $a_i(t) \to 0$ weakly in B_0 for every t, hence also strongly in B because the imbedding $B_0 \subset B$ is compact. Therefore also $|a_i(t)|_\iota^p \to 0$ because of the continuity of the imbedding $B \subset B_1$. Obviously, the sequence $\{v_i\}_{i \in N}$ is bounded in $L^p(0, T; B_0)$, hence also in $L^1(0, T; B)$, and we can estimate:

$$\|a_i(t)\|_B \leqq \frac{1}{s} \int_0^s \|v_i(t + \tau)\|_B \, \mathrm{d}\tau \leqq \frac{1}{s} \|v_i\|_{L^1(0,T;B)} .$$

Using again the continuity of the imbedding $B \subset B_1$, we see that also $|a_i(t)|_\iota^p$ is bounded (independently of t and i), and we can employ the Lebesgue theorem to show the convergence of $I_1 = 2^{p-1} \int_0^{T/2} |a_i(t)|_\iota^p \, \mathrm{d}t$ to 0 for $i \to \infty$. Altogether we have proved (4). ■

Let us consider the set $W^{p,q}(0, T; B_0, B_1)$ from (1) endoved with the collection of the (semi)norms $v \mapsto \|v\|_{L^p(0,T;B_0)}$ and $v \mapsto |dv/dt|_{q\iota}$, $\iota \in I$. It clearly makes $W^{p,q}(0, T; B_0, B_1)$ a locally convex space. Then the above theorem immediately offers a generalization of the Lions-Temam theorem.

Corollary. *Let the assumptions of Theorem above be fulfilled and, in addition, let B_0 be reflexive, $1 < p < +\infty$, and $1 \leqq q \leqq +\infty$. Then the imbedding $W^{p,q}(0, T; B_0, B_1) \subset L^p(0, T; B)$ is compact.*

Proof. As $L^p(0, T; B)$ is a metric space with the completion $L^p(0, T; \bar{B})$ (recall that $\bar{B}$ denotes the Banach space corresponding to B), we are only to show that every sequence $\{v_i\}_{i\in N}$, bounded in $W^{p,q}(0, T; B_0, B_1)$, contains a subsequence converging (strongly) in $L^p(0, T; \bar{B})$. Since B_0 is reflexive and $1 < p < +\infty$, $L^p(0, T; B_0)$ is reflexive as well, and thus there is a subsequence $\{v_{i_k}\}_{k\in N}$ converging weakly to some $v \in L^p(0, T; B_0)$. As the sequence $\{dv_{i_k}/dt\}_{k\in N}$ is bounded in $L^q(0, T; B_1)$, it is bounded in $L^1(0, T; B_1)$ as well. Thus we can use our theorem, which gives the strong convergence of $\{v_{i_k}\}_{k\in N}$ even in $L^p(0, T; B)$, hence in $L^p(0, T; \bar{B})$, too. ■

To outline some applications in numerical analysis we consider, as a simple model example, the nonlinear parabolic equation describing e.g. a Stefan problem in the so-called enthalpy formulation (the notation will be standard):

$$\frac{\partial z}{\partial t} = \Delta\beta(z) \quad \text{on} \quad \Omega \times (0, T)$$

with an initial condition $z(\cdot, 0) = z_0$ and the Dirichlet boundary condition $\beta(z(x, \cdot)) = 0$ for $x \in \partial\Omega$, where $\partial\Omega$ is the boundary of the Lipschitz domain Ω and $\beta: \mathbf{R} \to \mathbf{R}$ is a nondecreasing continuous function. An approximate solution $z_h \in L^2(0, T; V_h)$ obtained after a spatial discretization of a finite-element type ($h > 0$ denotes a mesh parameter) fulfils the identity:

$$(5) \qquad \left\langle \frac{\partial}{\partial t} z_h, v \right\rangle = \langle \nabla\beta(z_h), \nabla v \rangle$$

for all $v \in V_h$ and a.a. $t \in [0, T]$, where V_h is a finite-dimensional subspace of the Sobolev space $H_0^1(\Omega)$, and $\langle \cdot, \cdot \rangle$ is the standard scalar product in $L^2(\Omega)$. Typically, $V_{h_1} \subset V_{h_2}$ for $h_1 \geqq h_2 > 0$ and $\bigcup_{h>0} V_h$ is dense in $H_0^1(\Omega)$. Sometimes, e.g. if β^{-1} is not Lipschitz, we cannot estimate the time derivative of $\beta(z_h)$ and we are forced to estimate the time derivative of z_h. However, we cannot estimate it directly in the norm of $L^2(0, T; H^{-1}(\Omega))$ because we cannot test (5) by general functions $v \in H_0^1(\Omega)$. Nevertheless, putting $v = v(t) \in V_h$ with $\|v\|_{L^2(0,T;H^1(\Omega))} \leqq 1$ into (5) and integrating it over the time interval $[0, T]$, we can estimate (under some additional assumptions) $|\int_0^T \langle \partial z_h/\partial t, v \rangle \, dt| \leqq C$ with C independent of h. This yields the estimate of $\partial z_h/\partial t$ for every $h \leqq h_0$ in the seminorm $|\cdot|_{p\iota}$ with $p = 2$, $\iota = h_0$, and $|u|_{h_0} =$

$= \sup \{\langle u, v\rangle;\ v \in V_{h_0}, \|v\|_{H_0^1(\Omega)} \leqq 1\}$. As $\bigcup_{h>0} V_h$ is dense in $H_0^1(\Omega)$, the collection of the seminorms $\{|\cdot|_h\}_{h>0}$ generates a Hausdorff topology on $B_1 = H^{-1}(\Omega)$, hence our theorem can be readily employed with $B_0 = L^2(\Omega)$, $B = H^{-1}(\Omega)$, and $p = q = 2$.

Acknowledgement. The author is indebted to Professor J. Nečas for motivating comments in the course of preparing the paper.

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Souhrn

ZOBECNĚNÍ LIONS-TEMAMOVY VĚTY O KOMPAKTNÍM VNOŘENÍ

TOMÁŠ ROUBÍČEK

Známá věta J. L. Lionse a R. Temama o kompaktním vnoření prostoru $\{v \in L^p(0, T; B_0);\ \mathrm{d}v/\mathrm{d}t \in L^q(0, T; B_1)\}$ do $L^p(0, T; B)$ je zobecněna pro případ, kdy B_0 je reflexivní Banachův prostor, vnořený kompaktně do normovaného lineárního prostoru B, jenž je spojitě vnořen do Hausdorffova lokálně konvexního prostoru B_1, a $1 < p < +\infty$, $1 \leqq q \leqq +\infty$. Je naznačeno užití takového zobecnění v numerické analýze.

Резюме

ОБОБЩЕНИЕ ТЕОРЕМЫ ЛИОНСА-ТЕМАМА О КОМПАКТНОМ ВЛОЖЕНИИ

TOMÁŠ ROUBÍČEK

Известная теорема Ж. Л. Лионса и Р. Темана о компактном вложении пространства $\{v \in L^p(0, T; B_0);\ \mathrm{d}v/\mathrm{d}t \in L^q(0, T; B_1)\}$ в $L^p(0, T; B)$ обобщается для случая, когда B_0 рефлексивное банахово пространство, вложеное компактно в нормированное линейное пространство B, котороe вложено непрерывно в отделимое локально выпуклое пространство B_1, и $1 < p < +\infty$, $1 \leqq q \leqq +\infty$. Указывается применение таково обобщения в вычислительном анализе.

Author's address: Ústav teorie informace a automatizace ČSAV, Pod vodárenskou věží 4, CS—182 08 Praha 8.