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A GENERALIZATION OF THE LIONS-TEMAM COMPACT IMBEDDING THEOREM

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Summary. The well-known theorem by J. L. Lions and R. Temam concerning the compact imbedding of the space \( \{ v \in L^p(0, T; B_0); \frac{dv}{dt} \in L^q(0, T; B_1) \} \) into \( L^p(0, T; B) \) is generalized to the case when \( B_0 \) is a reflexive Banach space imbedded compactly into a normed linear space \( B \) that is continuously imbedded into a Hausdorff locally convex space \( B_1 \), and \( 1 < p < +\infty \), \( 1 \leq q \leq +\infty \). Applications of such generalization to numerical analysis are outlined.

Keywords: compact imbedding, evolution equations, locally convex spaces.

AMS Subject Classification: Primary 46A50, Secondary 35K65, 65Mxx.

In [1; Chap. I, Thm. 5.1] and [2; Chap. III, Thm. 2.1] J. L. Lions and R. Temam posed the broadly applicable theorem concerning the compact imbedding of the space

\[
W^{p,q}(0, T; B_0, B_1) = \left\{ v \in L^p(0, T; B_0); \frac{dv}{dt} \in L^q(0, T; B_1) \right\}
\]

into the space \( L^p(0, T; B) \), where \( B_0 \subset B \subset B_1 \) are three Banach spaces, \( B_0, B_1 \) are reflexive, the imbedding \( B_0 \subset B \) is compact and \( B \subset B_1 \) is continuous, \( 1 < p < +\infty \), \( 1 < q < +\infty \), and \( T > 0 \). This theorem is very powerful since \( B_1 \) can be chosen arbitrarily large. The aim of this short note is to show that, in fact, it is sufficient to take for \( B_1 \) even an arbitrary locally convex space with the only condition that its topology is a Hausdorff one. Besides, \( q \) may be equal to 1 or \( +\infty \) and \( B \) need not be complete. At the end of this note some applications of such generalization will be briefly outlined.

Let \( B_1 \) be a locally convex space, \( \{ \| \cdot \|_i \}_i \) being a collection of seminorms generating its topology (\( I \) is an index set). Let the seminorm \( \| \cdot \|_q \) be defined by

\[
\| v \|_q = \begin{cases} 
 (\int_0^T |v(t)|^q \, dt)^{1/q} & \text{if } 1 \leq q < +\infty \\
 \esssup_{0 \leq t \leq T} |v(t)|, & \text{if } q = +\infty 
\end{cases}
\]

Put \( L^q(0, T; B_1) = \{ v: [0, T] \to B_1; v \ \text{is Bochner integrable}, \| v \|_q < +\infty \ \forall t \in I \} \). By endowing \( L^q(0, T; B_1) \) with a collection of the seminorms \( \{ \| \cdot \|_q \}_q \), we obviously get a locally convex space. As usual, we will understand a linear operator to be compact if it maps bounded subsets into precompact ones.
Theorem. Let $B_0$ be a normed linear space imbedded compactly into another normed linear space $B$ which is continuously imbedded into a Hausdorff locally convex space $B_1$, and $1 \leq p < +\infty$. If $v$ in $L^p(0, T; B_0)$, $i \in \mathbb{N}$, the sequence $\{v_i\}_{i \in \mathbb{N}}$ converges weakly to $v$ in $L^p(0, T; B_0)$, and $\{dv_i/dt\}_{i \in \mathbb{N}}$ is bounded in $L^1(0, T; B_1)$, then $\{v_i\}_{i \in \mathbb{N}}$ converges to $v$ strongly in $L^p(0, T; B)$.

Proof. First we will prove that $\forall \eta > 0 \exists J_\eta \in \mathcal{F}(I) \exists c_\eta \in R \forall u \in B_0$:

(2) $\|u\|^p_B \leq \eta \|u\|^p_{B_0} + c_\eta \sum_{i \in J_\eta} |u|^p_i$,

where $\mathcal{F}(I)$ is the set of all finite subsets of $I$. Supposing the contrary, we get $\eta > 0$ such that $\forall J \in \mathcal{F}(I) \forall c \in R \exists u_J \in B_0$: $\|u_J\|^p_B \geq \eta \|u_J\|^p_{B_0} + c \sum_{i \in J} |u_i|^p_i$. Putting $w_J = u_J/\|u_J\|_{B_0}$, we get:

(3) $\|w_J\|^p_B \geq \eta + c \sum_{i \in J} |w_i|^p_i$,

and also $\|w_J\|_B \leq C$, where $C = \sup_{u \in B_0} \|u\|_B/\|u\|_{B_0}$ represents the norm of the imbedding operator $B_0 \to B$. Hence $\sum_{i \in J} |w_i|^p \leq C\|u\|^p/\|u\|_{B_0}$, and thus also $|w_i|^p \leq C^1/C$ whenever $i \in J$. Thus $\lim_{i \to +\infty, J \in \mathcal{F}(I)} |w_i| = 0$ for every $i \in I$. Note that $\mathcal{F}(I)$ and $R$ are directed by the relations $\subset$ and $\leq$, respectively, and thus we can speak actually about the net $\{w_i\}_{J \in \mathcal{F}(I), \epsilon \in R}$ and about its possible limit.

This net forms a precompact subset of $B$ because it is bounded in $B_0$ which is compactly imbedded into $B$. Hence there is its subnet (denote it by same indices, for simplicity) such that $w_{J_\epsilon} \to w$ strongly in $B$, where $B$ denotes the completion of $B_1$. As the imbedding $B_0 \subset B_1$ is continuous, each of the seminorms $|\cdot|_i$ is uniformly continuous on $B_0$, and we may extend it continuously on $B_0$, denoting the extension again by $|\cdot|_i$, for simplicity. Therefore we have $|w_{J_\epsilon} - w_i| \to 0$ for every $i \in I$. Clearly, we can write $|w_i| \leq |w_{J_\epsilon} - w_i| + |w_{J_\epsilon} - w_i|$. Passing to the limit, we get $|w_i| = 0$ for every $i \in I$. Thus $w = 0$ because the topology of $B_1$ has been supposed to be Hausdorff. In other words, $w_{J_\epsilon} \to 0$ strongly in $B$, which contradicts (3), thus proving (2).

Without loss of generality we may take $v = 0$. Let $\epsilon > 0$. As the sequence $\{v_i\}_{i \in \mathbb{N}}$ is bounded in $L^p(0, T; B_0)$, we can take $\eta = \epsilon/(2 \cdot \sup_{i \in \mathbb{N}} \|v_i\|^p_{L^p(0, T; B_0)})$. Integrating (2) over $[0, T]$ we get

(4) $\lim_{i \to +\infty} \|v_i\|^p_{L^p(0, T; B_0)} = 0$ for all $i \in I$.

Clearly, $\|v_i\|^p_{L^p(0, T; B_0)} = \int_0^T |v_i(t)|^p dt$ and we may investigate only the first term, while the second can be treated analogously. For every $s > 0$ such that $s \leq T/2$ and every $i \in [0, T/2]$ we may write $v_i(t) = a_i(t) + b_i(t)$, where

$$a_i(t) = \frac{1}{s} \int_0^s v_i(t + \tau) d\tau \quad \text{and} \quad b_i(t) = \int_0^s \left(\frac{\tau}{s} - 1\right) \frac{v_i(t + \tau)}{dt} d\tau.$$
Hence
\[ \int_0^{T/2} [v_i(t)]^p \, dt = 2^{p-1} \int_0^{T/2} |a_i(t)|^p \, dt + 2^{p-1} \int_0^{T/2} |b_i(t)|^p \, dt = I_1 + I_2. \]

We can estimate
\[ I_2 \leq 2^{p-1} \int_0^{T/2} \left( \int_0^s \left( 1 - \frac{\tau}{s} \right) \left| \frac{d}{dt} v_i(t + \tau) \right| \, d\tau \right)^p \, dt = \]
\[ = 2^{p-1} \left\| \left[ \frac{d}{dt} v_i \right] * \psi_s \right\|_{L_p(0,T/2)}^p, \]

where "\(*\)" denotes the convolution, i.e. \[ [f * g](t) = \int f(\tau) g(t - \tau) \, d\tau, \]
and \( \psi_s : R \to R \) is defined by
\[ \psi_s(t) = \begin{cases} t/s + 1 & \text{for } -s \leq t \leq 0, \\ 0 & \text{elsewhere.} \end{cases} \]

The following estimates are well-known: \( \| f * g \|_{L^1(R)} \leq \| f \|_{L^1(R)} \| g \|_{L^1(R)} \) and \( \| f * g \|_{L^\infty(R)} \leq \| f \|_{L^1(R)} \| g \|_{L^\infty(R)}. \) As \( g \mapsto f * g \) is a linear operator on \( L^1(R), \) we can obtain by interpolation (using the classical Riesz-Thorin convexity theorem) the estimate
\[ \| f * g \|_{L^p(R)} \leq \| f \|_{L^1(R)} \| g \|_{L^p(R)}. \]

It yields the estimate
\[ \left\| \left[ \frac{d}{dt} v_i \right] * \psi_s \right\|_{L_p(0,T/2)} \leq \left\| \frac{d}{dt} v_i \right\|_{L^1(0,T/2+s)} \| \psi_s \|_{L_p(R)}. \]

As \( \| \psi_s \|_{L_p(R)} \leq s^{1/p}, \) we get \( I_2 \leq 2^{p-1} s \| \left[ \frac{d}{dt} v_i \right] \|_{L^p(R)}^p, \) and we see that \( I_2 = O(s) \) for \( s \to 0 \) because, by the assumptions, \( \left\{ \frac{d}{dt} v_i \right\}_{i \in N} \) is bounded in \( L^1(0,T; B_0), \) hence particularly in the seminorm \( \| \cdot \|_{L^1} \). Thus the term \( I_2 \) can be made arbitrarily small when taking \( s \) small enough.

Now, let us take \( s > 0 \) fixed and investigate the term \( I_1. \) Since \( v_i \to 0 \) weakly in \( L^p(0,T; B_0), \) we can see that \( a_i(t) \to 0 \) weakly in \( B_0 \) for every \( t, \) hence also strongly in \( B \) because the imbedding \( B_0 \subset B \) is compact. Therefore also \( \| a_i(t) \|_p \to 0 \) because of the continuity of the imbedding \( B \subset B_1. \) Obviously, the sequence \( \left\{ v_i \right\}_{i \in N} \) is bounded in \( L^p(0,T; B_0), \) hence also in \( L^1(0,T; B), \) and we can estimate:
\[ \| a_i(t) \|_B \leq \frac{1}{s} \int_0^s \| v_i(t + \tau) \|_B \, d\tau \leq \frac{1}{s} \| v_i \|_{L^1(0,T; B)}. \]

Using again the continuity of the imbedding \( B \subset B_1, \) we see that also \( \| a_i(t) \|_p \) is bounded (independently of \( t \) and \( i), \) and we can employ the Lebesgue theorem to show the convergence of \( I_1 = 2^{p-1} \int_0^{T/2} |a_i(t)|^p \, dt \) to \( 0 \) for \( i \to \infty. \) Altogether we have proved (4).
Let us consider the set $W^{p,q}(0, T; B_0, B_1)$ from (1) endowed with the collection of the (semi)norms $v \mapsto \|v\|_{L^p(0,T;B_0)}$ and $v \mapsto |dv/dt|_{H^1(I_{tt})}$, $t \in I$. It clearly makes $W^{p,q}(0, T; B_0, B_1)$ a locally convex space. Then the above theorem immediately offers a generalization of the Lions-Temam theorem.

**Corollary.** Let the assumptions of Theorem above be fulfilled and, in addition, let $B_0$ be reflexive, $1 < p < +\infty$, and $1 \leq q \leq +\infty$. Then the imbedding $W^{p,q}(0, T; B_0, B_1) \subset L^p(0, T; B)$ is compact.

**Proof.** As $L^p(0, T; B)$ is a metric space with the completion $L^p(0, T; \overline{B})$ (recall that $\overline{B}$ denotes the Banach space corresponding to $B$), we are only to show that every sequence $\{v_{ik}\}_{k \in \mathbb{N}}$, bounded in $W^{p,q}(0, T; B_0, B_1)$, contains a subsequence converging (strongly) in $L^p(0, T; B)$. Since $B_0$ is reflexive and $1 < p < +\infty$, $L^p(0, T; B_0)$ is reflexive as well, and thus there is a subsequence $\{v_{ik}\}_{k \in \mathbb{N}}$ converging weakly to some $v \in L^p(0, T; \overline{B_0})$. As the sequence $\{dv_{ik}/dt\}_{k \in \mathbb{N}}$ is bounded in $L^q(0, T; B_1)$, it is bounded in $L^q(0, T; B_1)$ as well. Thus we can use our theorem, which gives the strong convergence of $\{v_{ik}\}_{k \in \mathbb{N}}$ even in $L^p(0, T; B)$, hence in $L^p(0, T; \overline{B})$, too. 

To outline some applications in numerical analysis we consider, as a simple model example, the nonlinear parabolic equation describing e.g. a Stefan problem in the so-called enthalpy formulation (the notation will be standard):

$$\frac{\partial z}{\partial t} = \Delta \beta(z) \quad \text{on} \quad \Omega \times (0, T)$$

with an initial condition $z(\cdot, 0) = z_0$ and the Dirichlet boundary condition $\beta(z(x, \cdot)) = 0$ for $x \in \partial \Omega$, where $\partial \Omega$ is the boundary of the Lipschitz domain $\Omega$ and $\beta: \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function. An approximate solution $z_h \in L^2(0, T; V_h)$ obtained after a spatial discretization of a finite-element type ($h > 0$ denotes a mesh parameter) fulfils the identity:

$$(\nabla \beta(z_h), \nabla v) = \int_0^T \langle \partial_t z_h, v \rangle dt$$

for all $v \in V_h$ and a.a. $t \in [0, T]$, where $V_h$ is a finite-dimensional subspace of the Sobolev space $H^1_0(\Omega)$, and $\langle \cdot, \cdot \rangle$ is the standard scalar product in $L^2(\Omega)$. Typically, $V_{h_1} \subset V_{h_2}$ for $h_1 \geq h_2 > 0$ and $\bigcup_{h > 0} V_h$ is dense in $H^1_0(\Omega)$. Sometimes, e.g. if $\beta^{-1}$ is not Lipschitz, we cannot estimate the time derivative of $\beta(z_h)$ and we are forced to estimate the time derivative of $z_h$. However, we cannot estimate it directly in the norm of $L^2(0, T; H^{-1}(\Omega))$ because we cannot test (5) by general functions $v \in H^1_0(\Omega)$. Nevertheless, putting $v = v(t) \in V_h$ with $\|v\|_{L^2(0,T;H^1(I_{tt}))} \leq 1$ into (5) and integrating it over the time interval $[0, T]$, we can estimate (under some additional assumptions) $\int_0^T \langle \partial_t z_h, v \rangle dt \leq C$ with $C$ independent of $h$. This yields the estimate of $\partial_z z_h/\partial t$ for every $h \leq h_0$ in the seminorm $\|\cdot\|_{p,t} = \|\cdot\|_{p_0} = \|\cdot\|_{h_0} = $
\[ \sup \{ \langle u, v \rangle ; v \in V_h, \| u \|_{H^1(\Omega)} \leq 1 \}. \] As \( \bigcup_{h>0} V_h \) is dense in \( H^1_0(\Omega) \), the collection of the seminorms \( \{ |\cdot|_{H^1_0} \}_{h>0} \) generates a Hausdorff topology on \( B_1 = H^{-1}(\Omega) \), hence our theorem can be readily employed with \( B_0 = L^2(\Omega), B = H^{-1}(\Omega), \) and \( p = q = 2 \).

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References


Souhrn

ZOBECNĚNÍ LIONS-TEMAMOVY VĚTY O KOMPAKTNÍM VNOŘENÍ

Tomáš Roubíček

Známá věta J. L. Lionse a R. Temama o kompaktním vnoření prostoru \( \{ v \in L^p(0, T; B_0); \text{dv}/\text{dt} \in L^q(0, T; B_1) \} \) do \( L^p(0, T; B) \) je zobecněna pro případ, kdy \( B_0 \) je reflexivní Banachův prostor, vnořený kompaktně do normovaného lineárního prostoru \( B \), jenž je spojitě vnořen do Hausdorffova lokálně konvexního prostoru \( B_1 \), a \( 1 < p < +\infty, 1 \leq q \leq +\infty \). Je naznačeno užití takového zobecnění v numerické analýze.

Резюме

ОБОБЩЕНИЕ ТЕОРЕМЫ ЛИОНСА-ТЕМАМА О КОМПАКТНОМ ВЛОЖЕНИИ

Томаш Рубичек

Известная теорема Ж. Л. Лионса и Р. Темана о компактном вложении пространства \( \{ v \in L^p(0, T; B_0); \text{dv}/\text{dt} \in L^q(0, T; B_1) \} \) в \( L^p(0, T; B) \) обобщается для случая, когда \( B_0 \) рефлексивное банахово пространство, вложенное компактно в нормированное линейное пространство \( B \), которое вложено непрерывно в оделимое локально выпуклое пространство \( B_1 \), и \( 1 < p < +\infty, 1 \leq q \leq +\infty \). Указывается применение такого обобщения в вычислительном анализе.

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