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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 32 (1991), No. 3, 441--446

Persistent URL: <http://dml.cz/dmlcz/118425>

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## Weak uniform rotundity of Musielak–Orlicz spaces

MALGORZATA DOMAN

*Abstract.* We give necessary and sufficient conditions for weak uniform rotundity of Musielak–Orlicz spaces  $L_\varphi$  with the Luxemburg norm. The result is a generalization of a theorem by Kamińska and Kurc.

*Keywords:* Musielak–Orlicz space, rotundity

*Classification:* 46B20, 46B25

### Introduction.

Let  $T$  be a set,  $\Sigma$  a  $\sigma$ -algebra of subsets of  $T$ ,  $\mu$  a non-negative atomless  $\sigma$ -finite complete measure on  $\Sigma$ . A function  $\varphi : R_+ \times T \rightarrow R_+$ , where  $R_+ = [0, +\infty)$ , is said to be a Musielak–Orlicz function if  $\varphi(0, t) = 0$  for  $\mu$ -almost every  $t \in T$ ,  $\varphi(\cdot, t)$  is a convex function on  $R_+$  for  $\mu$ -almost every  $t \in T$  and  $\varphi(u, \cdot)$  is a  $\Sigma$ -measurable function on  $T$  for every  $u \geq 0$ . The complementary function to a function  $\varphi$  is defined by  $\varphi^*(v, t) = \sup_{u > 0} (vu - \varphi(u, t))$  for  $v \geq 0, t \in T$ . We denote by  $M$  the set of all  $\Sigma$ -measurable functions  $x : T \rightarrow R$ . The functions which are different only on a null-set are considered as identical. The Musielak–Orlicz space  $L_\varphi$  is a subset of  $M$  such that  $I_\varphi(\lambda x) = \int_T \varphi(\lambda|x(t)|, t) d\mu < +\infty$  for some  $\lambda > 0$  dependent on  $x$ . The functionals  $\|x\|_\varphi = \inf\{r > 0 : I_\varphi(\frac{x}{r}) \leq 1\}$  and  $\|x\|_\varphi^0 = \sup\{\int_T x(t)y(t) d\mu : y \in L_{\varphi^*}, I_{\varphi^*}(y) \leq 1\}$  are norms in this space, called the Luxemburg and the Orlicz norm, respectively. We say that a function  $\varphi$  satisfies the condition  $\Delta_\alpha$ , for some  $\alpha > 1$ , if there are a constant  $K_\alpha > 0$  and a function  $h_\alpha : T \rightarrow R_+$ , such that  $\int_T h_\alpha(t) d\mu < +\infty$  and  $\varphi(\alpha u, t) \leq K_\alpha \varphi(u, t) + h_\alpha(t)$  for almost every  $t \in T$  and for  $u \geq u_0$  ( $u_0$ -some positive constant), when  $\mu(T) < +\infty$ , or for all  $u \in R_+$ , when  $\mu(T) = +\infty$ . Recall that a function  $\varphi$  is called strictly convex, if for all  $u, v \in R_+, u \neq v, \alpha, \beta \in R_+ \setminus \{0\}, \alpha + \beta = 1$ , we have  $\varphi(\alpha u + \beta v, t) < \alpha \varphi(u, t) + \beta \varphi(v, t)$  outside of some null-set. For further details concerning Musielak–Orlicz spaces see [7].

We say that a Banach space  $(X, \|\cdot\|)$  is weakly uniformly rotund (WUR), if for every  $x^* \in X, x^* \neq 0$  and  $\varepsilon > 0$  there exists  $\delta(x^*, \varepsilon) > 0$ , such that if  $\|x\| = \|y\| = 1$  and  $x^*(x - y) \geq \varepsilon$ , then  $\|\frac{x+y}{2}\| \leq 1 - \delta(x^*, \varepsilon)$  (cf. [1]). If for all  $x, y \in X$  such that  $\|x\| = \|y\| = 1$  we have  $\|\frac{x+y}{2}\| < 1$ , then we say that  $(X, \|\cdot\|)$  is rotund.

The aim of this paper is to give necessary and sufficient conditions for WUR of Musielak–Orlicz spaces. The result is a generalization of a theorem by Kamińska and Kurc ([6, Theorem 2.8]).

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I am greatly indebted to Professor Anna Kamińska for suggesting the problem discussed here to me. I wish to thank also Professor Henryk Hudzik for his help in preparing this paper

**Results.**

For the proof of the main theorem, we need some lemmas.

**Lemma 1** (cf. [6]). *If an arbitrary Banach space contains an isomorphic copy of  $l_1$ , then  $X$  is not WUR.*

**Lemma 2.** *If  $\varphi$  is a strictly convex Musielak–Orlicz function, then for every  $\varepsilon > 0$  and every  $\Sigma$ -measurable functions  $p, q : T \rightarrow (0, +\infty), p(t) < q(t)$  for  $\mu$ -almost every  $t \in T$ , there exists a  $\Sigma$ -measurable function  $r : T \rightarrow (0, 1)$  such that*

$$\varphi\left(\frac{u+v}{2}, t\right) \leq \frac{1-r(t)}{2}\{\varphi(u, t) + \varphi(v, t)\}$$

for  $\mu$ -almost every  $t \in T$  whenever  $|u - v| \geq \varepsilon \max\{|u|, |v|\}$  and  $\max\{|u|, |v|\} \in [p(t), q(t)]$ .

The proof of this lemma is analogous to that of Lemma 1 in [5], so it is omitted here. □

**Lemma 3.** *Assume that  $\varphi$  is a Musielak–Orlicz function satisfying the  $\Delta_2$ -condition. Then for every  $\alpha > 1$  and  $\varepsilon > 0$ , there is a set  $T_0$  of measure 0, a constant  $K_{\alpha,\varepsilon} > 0$  and a  $\Sigma$ -measurable function  $h_{\alpha,\varepsilon} : T \rightarrow [0, +\infty)$  such that  $\int_T h_{\alpha,\varepsilon}(t) d\mu \leq \varepsilon$  and  $\varphi(\alpha u, t) \leq K_{\alpha,\varepsilon}\varphi(u, t) + h_{\alpha,\varepsilon}(t)$  for any  $t \in T \setminus T_0$  and any  $u \in R$ .*

The proof for  $\alpha = 2$  is given in [4]. The proof for an arbitrary  $\alpha > 1$  can proceed in the same way, if we notice that  $\varphi$  satisfies the  $\Delta_2$ -condition if and only if it satisfies the  $\Delta_\alpha$ -condition for every  $\alpha > 1$ . □

**Lemma 4** (cf. [4]). *Let  $\varphi$  be a Musielak–Orlicz function that satisfies the  $\Delta_2$ -condition. Then*

- (i) *there is a function  $\beta : (0, 1) \rightarrow (0, 1)$  such that  $\|x\|_\varphi \leq 1 - \beta(\varepsilon)$  whenever  $I_\varphi(x) \leq 1 - \varepsilon$ .*
- (ii)  *$\|x\|_\varphi = 1$  if and only if  $I_\varphi(x) = 1$ .*

**Lemma 5.** *Assume that  $\varphi$  is a Musielak–Orlicz function vanishing only at 0 and that  $\varphi$  and  $\varphi^*$  satisfy the  $\Delta_2$ -condition. Let  $x^* \in (L_\varphi)^*$  be regular and nontrivial (i.e. there exists  $z \in L_{\varphi^*}, z \neq 0$  such that  $x^*(x) = \int_T x(t)z(t) d\mu$  for every  $x \in L_\varphi$ ). Let  $(B_n)_{n=1}^\infty$  be an increasing sequence of sets with finite and positive measures such that  $\bigcup_n B_n = \text{supp } z$ . Denote  $C_n = \{t \in T : \frac{1}{n} \leq |z(t)| \leq n\}$  and put  $D_n = C_n \cap B_n$ . Then  $(D_n)_{n=1}^\infty$  is increasing,  $\bigcup_n D_n = \text{supp } z$  and*

$$\int_{D_n} y(t)z(t) d\mu \rightarrow \int_T y(t)z(t) d\mu$$

uniformly with respect to  $y$  in every bounded set in  $L_\varphi$ .

PROOF: In virtue of B. Levi theorem and the  $\Delta_2$ -condition for  $\varphi^*$ , we have  $\|z - z_n\|_{\varphi^*}^0 \rightarrow 0$  as  $n \rightarrow +\infty$ , where  $z_n = z\chi_{D_n}$ . Then

$$\begin{aligned} 0 &\leq \left| \int_T y(t)z(t) \, d\mu - \int_{D_m} y(t)z(t) \, d\mu \right| = \\ &= \left| \int_T y(t)z(t) \, d\mu - \int_T y(t)z_m(t) \, d\mu \right| \leq \\ &\leq \|y\|_{\varphi} \|z - z_m\|_{\varphi^*}^0 \leq C \|z - z_m\|_{\varphi^*}^0. \end{aligned}$$

Hence the desired result follows. □

The next two lemmas are analogs of Lemma 2.5 and Lemma 2.6 of [6].

**Lemma 6.** *Let  $\mu(T) < +\infty$  and  $\varphi$  be a Musielak–Orlicz function such that for every  $t \in T$   $\frac{\varphi(u,t)}{u} \rightarrow +\infty$  as  $u \rightarrow +\infty$ . Then for every  $\varepsilon > 0$ , there exist  $\Sigma$ -measurable functions  $p, q : T \rightarrow (0, +\infty)$  such that for every  $x, y \in L_{\varphi}$  satisfying  $I_{\varphi}(x) = I_{\varphi}(y) = 1$  and  $\int_T |x(t) - y(t)| \, d\mu \geq \varepsilon$ , we have  $\int_A |x(t) - y(t)| \, d\mu \geq \frac{\varepsilon}{4}$  whenever*

$$A = \{t \in T : p(t) \leq \max(|x(t)|, |y(t)|) \leq q(t)\}.$$

PROOF: Define for any  $n \in N$   $p_n(t) = \inf\{u > 0 : \frac{\varphi(u,t)}{u} \geq n\}$ . Then  $p_n$  is a  $\Sigma$ -measurable function and  $\varphi(u,t) \geq nu$  for every  $u \geq p_n(t)$ . Define  $A_n = \{t \in T : |x(t)| \leq p_n(t)\}$ ,  $A_n^1 = \{t \in T : |y(t)| \leq p_n(t)\}$ . We have

$$\int_{T \setminus A_n} |x(t)| \, d\mu \leq \frac{1}{n} \int_{T \setminus A_n} \varphi(|x(t)|, t) \, d\mu \leq \frac{1}{n}.$$

In the same way, we can obtain  $\int_{T \setminus A_n^1} |y(t)| \, d\mu \leq \frac{1}{n}$ . Moreover,

$$\begin{aligned} \int_{T \setminus A_n} |y(t)| \, d\mu &= \int_{(T \setminus A_n) \cap (T \setminus A_n^1)} |y(t)| \, d\mu + \int_{A_n^1 \setminus A_n} |y(t)| \, d\mu \leq \\ &\leq \int_{T \setminus A_n^1} |y(t)| \, d\mu + \int_{T \setminus A_n} |x(t)| \, d\mu \leq \frac{2}{n}. \end{aligned}$$

Similarly  $\int_{T \setminus A_n^1} |x(t)| \, d\mu \leq \frac{2}{n}$ . Hence  $\int_{T \setminus (A_n \cap A_n^1)} |x(t) - y(t)| \, d\mu \leq \int_{T \setminus A_n} |x(t)| \, d\mu + \int_{T \setminus A_n^1} |y(t)| \, d\mu + \int_{T \setminus A_n^1} |x(t)| \, d\mu + \int_{T \setminus A_n} |y(t)| \, d\mu \leq \frac{6}{n}$ . Since  $\int_T |x(t) - y(t)| \, d\mu \geq \varepsilon$  by the assumptions, we have  $\int_{A_n \cap A_n^1} |x(t) - y(t)| \, d\mu \geq \varepsilon - \frac{6}{n} \geq \frac{\varepsilon}{2}$  if  $n$  is such that  $\frac{6}{n} \leq \frac{\varepsilon}{2}$ . Define  $A_n^2 = \{t \in T : \frac{\varepsilon}{8\mu(T)} \leq \max(|x(t)|, |y(t)|)\}$ . If  $t \notin A_n^2$ , then  $|x(t)| < \frac{\varepsilon}{8\mu(T)}$  and  $|y(t)| < \frac{\varepsilon}{8\mu(T)}$ . Therefore  $\int_{(A_n \cap A_n^1) \setminus A_n^2} |x(t) - y(t)| \, d\mu \leq \frac{\varepsilon}{8\mu(T)} \mu(T \setminus A_n^2) + \frac{\varepsilon}{8\mu(T)} \mu(T \setminus A_n^2) \leq \frac{\varepsilon}{4}$ . Thus  $\int_{A_n \cap A_n^1 \cap A_n^2} |x(t) - y(t)| \, d\mu \geq \frac{\varepsilon}{2} - \frac{\varepsilon}{4} = \frac{\varepsilon}{4}$ . Putting  $A = A_n \cap A_n^1 \cap A_n^2$ ,  $p(t) = \frac{\varepsilon}{8\mu(T)}$  and  $q(t) = p_n(t)$ , we get the desired inequality. □

**Lemma 7.** *Let  $\varphi$  be a Musielak–Orlicz function satisfying the  $\Delta_2$ -condition and let  $B \in \Sigma, \varepsilon > 0$  and  $\sigma \in (0, 1)$  be such that  $I_\varphi((x - y)\chi_B) \geq \varepsilon$  and  $I_\varphi(\frac{x+y}{2}) \leq 1 - \frac{\sigma}{2}(I_\varphi(x\chi_B) + I_\varphi(y\chi_B))$ , where  $x, y$  are arbitrary measurable functions with  $I_\varphi(x) = I_\varphi(y) = 1$ . Then there exists a constant  $q \in (0, 1)$ , such that  $I_\varphi(\frac{x+y}{2}) \leq 1 - q$ .*

PROOF: Let  $K = K_{2, \frac{\varepsilon}{2}}$ , where  $K_{2, \frac{\varepsilon}{2}}$  is the constant from Lemma 3. Then

$$\varepsilon \leq I_\varphi((x - y)\chi_B) \leq \frac{K}{2}(I_\varphi(x\chi_B) + I_\varphi(y\chi_B)) + \frac{\varepsilon}{2}.$$

Hence  $I_\varphi(x\chi_B) + I_\varphi(y\chi_B) \geq \frac{\varepsilon}{2} \cdot \frac{2}{K} = \frac{\varepsilon}{K}$ . Therefore  $I_\varphi(\frac{x+y}{2}) \leq 1 - \frac{\sigma\varepsilon}{2K}$ , and it suffices to put  $q = \frac{\sigma\varepsilon}{2K}$  □

**Theorem 1.** *A Musielak–Orlicz space  $L_\varphi$  is WUR if and only if*

- (i)  $\varphi$  is strictly convex,
- (ii)  $\varphi$  satisfies the  $\Delta_2$ -condition,
- (iii)  $\varphi^*$  satisfies the  $\Delta_2$ -condition.

PROOF: Sufficiency. Assume that the conditions (i), (ii), (iii) are satisfied. Let  $x, y \in L_\varphi, \|x\|_\varphi = \|y\|_\varphi = 1, x^* \in (L_\varphi)^*$  and  $x^*(x - y) \geq \varepsilon$ , where  $\varepsilon \in (0, 1)$ . In virtue of the representation of  $x^*$ , we have  $\int_T(x(t) - y(t))z(t) d\mu \geq \varepsilon$  for some  $z \in L_{\varphi^*}$ . Define  $z_n$  as in the proof of Lemma 5. Then in view of this lemma, there is  $n_0 \in N$  ( $n_0$  independent of  $x$  and  $y$ ) such that  $\int_T(x(t) - y(t))z_{n_0}(t) d\mu \geq \frac{\varepsilon}{2}$ . Since  $|z_{n_0}(t)| < n_0$ , denoting  $T_0 = \text{supp } z_{n_0}$ , we get  $\int_{T_0} |x(t) - y(t)| d\mu \geq \frac{\varepsilon}{2n_0}$ . Since, according to Lemma 2.4 of [6], (iii) implies  $\varphi(u, t)/u \rightarrow +\infty$  when  $u \rightarrow +\infty$  for every  $t \in T$ , it follows from Lemma 6 that there exist two  $\Sigma$ -measurable functions  $p, q : T_0 \rightarrow (0, +\infty)$ , such that denoting

$$A = \{t \in T_0 : p(t) \leq \max(|x(t), y(t)|) \leq q(t)\}, \text{ we have}$$

$$\int_A |x(t) - y(t)| d\mu \geq \frac{\varepsilon}{8n_0}.$$

Define  $B = \{t \in A : |x(t) - y(t)| \geq \frac{\varepsilon}{8n_0K} \max(|x(t)|, |y(t)|)\}$ , where  $K = K_{\alpha, \frac{1}{2}}$  is the constant from Lemma 3 corresponding to  $\alpha = \max\{\frac{64n_0}{\varepsilon} \|\chi_{T_0}\|_{\varphi^*}, 1\}$ . In virtue of Lemma 2 there is a function  $r : B \rightarrow (0, 1)$  such that

$$\varphi\left(\frac{|x(t) + y(t)|}{2}, t\right) \leq \frac{1 - r(t)}{2} \{\varphi(|x(t)|, t) + \varphi(|y(t)|, t)\}.$$

Define  $B_m = \{t \in B : r(t) \geq \frac{1}{m}\}$ . We have  $B_m \nearrow$  and  $\bigcup_{n=1}^\infty B_m = B$ . Thus, defining  $C_m = (A \setminus B) \cup B_m$ , we obtain the increasing sequence of sets such that  $\bigcup_{n=1}^\infty C_n = A$ . By Lemma 5 there is  $s \in N$  ( $s$  independent of  $x$  and  $y$ ) such that

$$\int_{C_s} |x(t) - y(t)| d\mu \geq \int_A |x(t) - y(t)| d\mu - \frac{1}{4} \cdot \frac{\varepsilon}{8n_0}.$$

i.e.

$$(1) \quad \int_{C_s} |x(t) - y(t)| d\mu \geq \frac{\varepsilon}{32n_0}.$$

For  $t \in B_s$ , we have

$$\varphi\left(\frac{|x(t) + y(t)|}{2}, t\right) \leq \frac{1 - \frac{1}{s}}{2} \{\varphi(|x(t)|, t) + \varphi(|y(t)|, t)\}.$$

Hence, using the convexity of  $\varphi$  and the fact that  $I_\varphi(x) = I_\varphi(y) = 1$ , we get

$$(2) \quad I_\varphi\left(\frac{x+y}{2}\right) \leq 1 - \frac{1}{2s} \{I_\varphi(x)\chi_{B_s} + I_\varphi(y)\chi_{B_s}\}.$$

If  $t \in A \setminus B$ , then

$$|x(t) - y(t)| < \frac{\varepsilon}{8n_0K} \max(|x(t)|, |y(t)|).$$

Hence

$$(3) \quad I_\varphi((x-y)\chi_{A \setminus B}) \leq \frac{\varepsilon}{8n_0K} \{I_\varphi(x\chi_{A \setminus B}) + I_\varphi(y\chi_{A \setminus B})\} \leq \frac{\varepsilon}{4n_0K}.$$

Applying the inequality (1) and the Hölder inequality, we get

$$2\|(x-y)\chi_{C_s}\|_\varphi \|\chi_{T_0}\|_{\varphi^*} \geq \int_{C_s} |x(t) - y(t)| d\mu \geq \frac{\varepsilon}{32n_0},$$

i.e.

$$\frac{64n_0}{\varepsilon} \|\chi_{T_0}\|_{\varphi^*} \|(x-y)\chi_{C_s}\|_\varphi \geq 1,$$

hence denoting  $\alpha_1 = \frac{64n_0}{\varepsilon} \|\chi_{T_0}\|_{\varphi^*}$ , we have  $\alpha_1 \leq \alpha$ , and

$$1 \leq I_\varphi(\alpha(x-y)\chi_{C_s}) \leq KI_\varphi((x-y)\chi_{C_s}) + \frac{1}{2}.$$

Thus

$$I_\varphi((x-y)\chi_{C_s}) \geq \frac{1}{2K}.$$

Combining this with the inequality (3), we get

$$I_\varphi((x-y)\chi_{B_s}) \geq I_\varphi((x-y)\chi_{C_s}) - I_\varphi((x-y)\chi_{A \setminus B}) \geq \frac{1}{2K} - \frac{\varepsilon}{4n_0K} = \beta.$$

Applying Lemma 7, the inequality (2) and the last inequality, we get

$$I_\varphi\left(\frac{x+y}{2}\right) \leq 1 - q.$$

Now, in view of Lemma 4, we have

$$\left\| \frac{x+y}{2} \right\|_{\varphi} \leq 1 - \beta(q),$$

where  $\beta(q) \in (0, 1)$ , and depends only on  $x^*$ ,  $\varepsilon$  and  $\varphi$ .

Necessity. If  $\varphi$  does not satisfy the condition (i) or the condition (ii), then  $L_{\varphi}$  is not rotund (cf. [5]). Since WUR implies rotundity,  $L_{\varphi}$  is not WUR as well. Assume now that  $\varphi$  satisfies the condition (i) and it does not satisfy the condition (iii). Then  $(L_{\varphi})^* = L_{\varphi^*}$ , where  $L_{\varphi^*}$  is equipped with the Orlicz norm. Since  $\varphi^*$  does not satisfy the  $\Delta_2$ -condition,  $L_{\varphi^*}$  contains an isomorphic copy of  $l_{\infty}$ . Hence it follows that  $L_{\varphi}$  contains an isomorphic copy of  $l_1$ . Therefore, in view of Lemma 1,  $L_{\varphi}$  is not WUR. The proof is finished.  $\square$

Theorem 1.2 of [3] and Theorem 1.2 of [2] imply the following version of our result.

**Theorem 2.** *A Musielak–Orlicz space  $L_{\varphi}$  is WUR if and only if it is rotund and reflexive.*

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(Received November 28, 1990, revised February 26, 1991)