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Abstract. The fundamental properties of approximate inverse systems of uniform spaces are established. The limit space of an approximate inverse sequence of complete metric spaces is the limit of an inverse sequence of some of these spaces. This has an application to the dimension of the limit space of an approximate inverse system. A topologically complete space with dim $\leq n$ is the limit of an approximate inverse system of metric polyhedra of dim $\leq n$. A completely metrizable separable space with dim $\leq n$ is the limit of an inverse sequence of locally finite polyhedra of dim $\leq n$. Finally, a new proof is derived of the important equality dim = Ind for metric spaces.

Keywords: inverse systems, approximate inverse systems, uniform, metric and complete spaces, covering and inductive dimension

Classification: 54B25, 54E15, 54F45

1. Introduction.

Mardešić and Rubin [9] have recently introduced and studied approximate inverse systems (AIS) of compact metric spaces. They proved that every compact Hausdorff space with dim $\leq n$ is the limit of an AIS consisting of finite polyhedra with dim $\leq n$. In contrast to this result, it has been known for some time that there are compact Hausdorff spaces of covering dimension 1 which are not homeomorphic to the limit space of an inverse system (IS) of polyhedra with dimension ≤ 1 [8], [11]. An AIS readily makes sense in the context of uniform spaces, and some obvious questions present themselves. For example, which properties of these spaces does the limit space inherit, and what spaces can be obtained as limit spaces of an AIS of "nice" (e.g. metric) spaces. Motivated by questions like these, in Section 2, we look for those properties of an AIS that correspond to the well-known fundamental results concerning an IS. We show, for example, that the limit space of an AIS is always a closed subspace of the product of the spaces making up the AIS. In fact, for all results in this paper, only one of the original three axioms of Mardešić and Rubin is needed, and the definition is weakened accordingly. In Section 3, we aim to prove that if an AIS consists of complete metric spaces with dim $\leq n$, then the limit space has dim $\leq n$. The purpose of Section 4 is to point out that the result of Mardešić and Rubin for compact spaces quoted above is generalized both to the class of topologically complete and the class of strongly paracompact Hausdorff spaces. A corollary is that a completely metrizable separable space with dim $\leq n$ is the limit space of an inverse sequence of locally finite polyhedra with dim $\leq n$.

Finally, in Section 5, we give a new proof of the equality dim = Ind for metric spaces. This transparent proof follows readily from results on inverse limits.

In this paper, uniform spaces are not necessarily Hausdorff, and $\dim X$ denotes the covering dimension of X defined in terms of cozero covers rather than open covers. The reader is referred to [3], [4], [6], [10], [13] for the facts of Dimension Theory and General Topology. The results in polyhedra needed in this paper can be found in [7], [10], [13], [14].

2. Definitions and basic results.

An AIS of uniform spaces $((X_\alpha, \mathcal{U}_\alpha), p_{\alpha\beta}, A)$ consists of a directed set A, a uniform space $(X_\alpha, \mathcal{U}_\alpha)$ for each $\alpha \in A$, and, for $\alpha < \beta$,a uniformly continuous function $p_{\alpha\beta} : X_\beta \to X_\alpha$ satisfying the following condition.

(AIS) For each $\alpha \in A$ and $U \in \mathcal{U}_\alpha$, there is in A $\alpha' > \alpha$ such that for $\alpha' < \beta < \gamma$, $|p_{\alpha\beta}p_{\beta\gamma} - p_{\alpha\gamma}| < U$, i.e. $(p_{\alpha\beta}p_{\beta\gamma}(x), p_{\alpha\gamma}(x)) \in U$ for all $x \in X_\gamma$.

Remark 1. In addition to the above condition, Mardešić and Rubin require their AIS to satisfy two more axioms that we dispense with. One drawback of the extra two axioms is that it is not immediately clear whether they are satisfied by any IS.

Remark 2. Entourages are taken to be symmetric so that $|x-y| < U$ iff $|y-x| < U$. Also the partial order $<$ on A is assumed to be anti-reflexive, i.e. $\alpha < \beta \Rightarrow \alpha \neq \beta$.

Throughout this section, we will be considering a fixed AIS $((X_\alpha, \mathcal{U}_\alpha), p_{\alpha\beta}, A)$. Its limit space X is the subspace of the product $\prod_{\alpha \in A} X_\alpha$ consisting of all points $x = (x_\alpha)$ such that, for each $\alpha \in A$, x_α is a limit point of the net $\{p_{\alpha\beta}(x_\alpha) : \alpha < \beta\}$, i.e.

(L) For each $\alpha \in A$ and $U \in \mathcal{U}_\alpha$, there is (in A) $\alpha' > \alpha$ such that for $\alpha' < \beta$, $|x_\alpha - p_{\alpha\beta}(x_\beta)| < U$.

At this point, we digress to deal with an obvious question that arises. Suppose our AIS is in fact an IS. Then its limit space X^* consists of all $x = (x_\alpha)$ with $x_\alpha = p_{\alpha\beta}(x_\beta)$ for $\alpha < \beta$. Obviously, $X^* \subset X$. We show that $X^* = X$ if each X_α is Hausdorff, and that otherwise the equality may fail.

Proposition 1. *If* $((X_\alpha, \mathcal{U}_\alpha), p_{\alpha\beta}, A)$ *is an IS and each* X_α *is Hausdorff, then* $X^* = X$ (*cf.* [9, Proposition 1]).

Proof: Let $x = (x_\alpha) \in X$ and $\alpha < \beta$. Since the limits of nets are unique in Hausdorff spaces,

$$x_\beta = \lim_{\beta < \gamma} p_{\beta\gamma}(x_\gamma).$$

Now, by continuity of $p_{\alpha\beta}$,

$$p_{\alpha\beta}(x_\beta) = \lim_{\beta<\gamma} p_{\alpha\beta}p_{\beta\gamma}(x_\gamma) = \lim_{\beta<\gamma} p_{\alpha\gamma}(x_\gamma) = \lim_{\alpha<\gamma} p_{\alpha\gamma}(x_\gamma) = x_\alpha\,.$$

Hence $x \in X^*$ and $X^* = X$. □

Example 1. For each $n \in N$, the set of positive integers, let X_n consists of at least one point x_n, and let $\mathcal{U}_n$ consists of $X_n \times X_n$. For $n < m$, let $p_{nm} : X_m \to X_n$ be given by $p_{nm}(x) = x_n$. We have an inverse sequence $((X_n, \mathcal{U}_n), p_{nm}, N)$ with X^* consisting of a single point (x_n) while $X = \prod_{n \in N} X_n$.

Returning to our AIS $((X_\alpha, \mathcal{U}_\alpha), p_{\alpha\beta}, A)$, the restriction of the canonical projection $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$ to X will be denoted by p_α.

The following result is the cornerstone of this section.

Proposition 2. *For $\alpha \in A$, and $U \in \mathcal{U}_\alpha$, there is $\alpha' > \alpha$ such that for $\alpha' < \beta$, $|p_\alpha - p_{\alpha\beta} p_\beta| < U$ (cf. [9, Lemma 4]).*

Proof: Let $x = (x_\alpha) \in X$ and pick $V \in \mathcal{U}_\alpha$ with $3V = V \circ V \circ V \subset U$. By (AIS), there is $\alpha' > \alpha$ such that for $\alpha' < \beta < \gamma$,

$$|p_{\alpha\gamma} - p_{\alpha\beta} p_{\beta\gamma}| < V. \tag{1}$$

For a fixed $\beta > \alpha'$, by the uniform continuity of $p_{\alpha\beta}$, there exists W in $\mathcal{U}_\beta$ such that

$$|y - z| < W \Rightarrow |p_{\alpha\beta}(y) - p_{\alpha\beta}(z)| < V. \tag{2}$$

Next, by (L), there is $\gamma > \beta$ such that

$$|x_\beta - p_{\beta\gamma}(x_\gamma)| < W \tag{3}$$

$$\text{and} \quad |x_\alpha - p_{\alpha\gamma}(x_\gamma)| < V. \tag{4}$$

Now by (2) and (3),

$$|p_{\alpha\beta} p_{\beta\gamma}(x_\gamma) - p_{\alpha\beta}(x_\beta)| < V. \tag{5}$$

Finally, by (4), (1) and (5),

$$|x_\alpha - p_{\alpha\beta}(x_\beta)| < 3V \subset U.$$

Hence, $|p_\alpha - p_{\alpha\beta} p_\beta| < U$. □

The next two results supply the necessary information about the uniformity of X.

Proposition 3. *Given $\alpha_i \in A$ and $U_i \in \mathcal{U}_{\alpha_i}$, $i = 1, 2, \dots, n$, there is $\alpha' \in A$ such that $\alpha' >$ each α_i and for $\alpha' < \alpha$, there is $U_\alpha \in \mathcal{U}_\alpha$ such that*

$$|p_\alpha(x) - p_\alpha(y)| < U_\alpha \Rightarrow |p_{\alpha_i}(x) - p_{\alpha_i}(y)| < U_i.$$

Proof: Choose $V_i \in \mathcal{U}_{\alpha_i}$ with $3V_i \subset U_i$. By Proposition 2, there is $\alpha' >$ each α_i such that for $\alpha > \alpha'$ and each i,

$$|p_{\alpha_i} - p_{\alpha_i \alpha} p_\alpha| < V_i. \tag{1}$$

For each $\alpha > \alpha'$, by the uniform continuity of $p_{\alpha_i\alpha}$, there is $U_\alpha \in \mathcal{U}_\alpha$ such that for each i,

$$(2) \qquad |x - y| < U_\alpha \Rightarrow |p_{\alpha_i\alpha}(x) - p_{\alpha_i\alpha}(y)| < V_i\,.$$

Now, if $|p_\alpha(x) - p_\alpha(y)| < U_\alpha$, then for each i, by (1), (2), (1), respectively,

$$\begin{aligned} &|p_{\alpha_i}(x) - p_{\alpha_i\alpha}p_\alpha(x)| < V_i,\\ &|p_{\alpha_i\alpha}p_\alpha(x) - p_{\alpha_i\alpha}p_\alpha(y)| < V_i,\\ \text{and} \quad &|p_{\alpha_i\alpha}p_\alpha(y) - p_{\alpha_i}(y)| < V_i\,. \end{aligned}$$

Hence

$$|p_{\alpha_i}(x) - p_{\alpha_i}(y)| < 3V_i \subset U_i\,.$$

□

Proposition 4. $\{(p_\alpha \times p_\alpha)^{-1}(U) : \alpha \in A',\, U \in \mathcal{U}_\alpha\}$ *is a base for the uniformity on X for any cofinal subset A' of A.*

Proof: Sets of the form $\bigcap_{i=1}^n (\pi_{\alpha_i} \times \pi_{\alpha_i})^{-1}(U_i)$ form a base for the uniformity on $\prod_{\alpha\in A} X_\alpha$. By Proposition 3, the intersection of any such set with $X \times X$ contains $(p_\alpha \times p_\alpha)^{-1}(U)$ for some $\alpha \in A'$ and $U \in \mathcal{U}_\alpha$, and the result follows. □

Our next result gives a property of X that is not always enjoyed by the limit space of an IS if it consists of non-Hausdorff spaces.

Proposition 5. *X is closed in $\prod_{\alpha\in A} X_\alpha$.*

Proof: Let $y \in Y = \prod_{\alpha\in A} X_\alpha - X$. Then, by (L), for some $\alpha \in A$ and $U \in \mathcal{U}_\alpha$, the set of $\beta \in A$ that satisfy

$$(1) \qquad |y_\alpha - p_{\alpha\beta}(y_\beta)| \not< U$$

is cofinal in A. Pick $V \in \mathcal{U}_\alpha$ with $3V \subset U$. In view of Proposition 2, there is $\beta > \alpha$ satisfying (1) and

$$(2) \qquad |p_\alpha - p_{\alpha\beta}p_\beta| < V.$$

Next, by the uniform continuity of $p_{\alpha\beta}$, there is W in $\mathcal{U}_\beta$ such that

$$(3) \qquad |x - y| < W \Rightarrow |p_{\alpha\beta}(x) - p_{\alpha\beta}(y)| < V.$$

Now, suppose there is a point $x \in X$ that satisfies

$$(4) \qquad |y - x| < (\pi_\alpha \times \pi_\alpha)^{-1}(V) \cap (\pi_\beta \times \pi_\beta)^{-1}(W).$$

$$\begin{aligned} &(5) \qquad \text{Then} \quad |y_\alpha - x_\alpha| < V\\ &(6) \qquad \text{and} \quad |y_\beta - x_\beta| < W. \end{aligned}$$

By (3) and (6),

$$|p_{\alpha\beta}(x_\beta) - p_{\alpha\beta}(y_\beta)| < V \tag{7}$$

and by (2),

$$|x_\alpha - p_{\alpha\beta}(x_\beta)| < V. \tag{8}$$

Finally, (5), (8) and (7) imply

$$|y_\alpha - p_{\alpha\beta}(y_\beta)| < 3V \subset U,$$

which contradicts the choice of β to satisfy (1). Thus, there is no x in X that satisfies (4), which shows that Y is open and hence X is closed in $\prod_{\alpha\in A} X_\alpha$. □

Some conclusions may now be readily drawn.

Corollary 1. *If each X_α is compact or complete, then the same is true of X.*

Corollary 2. *A topological space is the limit space of an AIS of metric spaces iff it is topologically complete.*

We next give a universal property of X which is readily seen to characterize X in the same fashion as the well known analogous result for inverse limits.

Proposition 6. *Let Y be a uniform space with the property that for each $\alpha \in A$, there is a uniformly continuous $f_\alpha : Y \to X_\alpha$ such that for any $U \in \mathcal{U}_\alpha$, there is $\alpha' > \alpha$ with $|f_\alpha - p_{\alpha\beta} f_\beta| < U$ for $\alpha' < \beta$. Then there is a unique uniformly continuous $f : Y \to X$ such that $f_\alpha = p_\alpha f$.*

Proof: It suffices to observe that $f(y) = (f_\alpha(y))$ is indeed a point of X for each $y \in Y$. □

With reference to our next and final result of this section, we recall that a subsystem of an IS over a cofinal subset has the same limit space. For an AIS, this always happens only if each space of the system is both Hausdorff and complete. The following two examples illustrate what might happen if these conditions are not present.

Proposition 7. *Let A' be a cofinal subset of A and X' the limit space of $((X_\alpha, \mathcal{U}_\alpha), p_{\alpha\beta}, A')$, and assume that each X_α is Hausdorff. Then there is an isomorphism $\pi : X \to X'$ into X' which is onto if additionally each X_α is complete.*

Proof: Let $\pi : X \to X'$ and $\pi'_\alpha : X' \to X_\alpha, \alpha \in A'$, be respectively the restrictions of the canonical projections $\prod_{\alpha\in A} X_\alpha \to \prod_{\alpha\in A'} X_\alpha$ and $\prod_{\alpha\in A'} X_\alpha \to X_\alpha$. By Proposition 4 and the fact that $p_\alpha = \pi'_\alpha \pi$ for $\alpha \in A'$, any entourage of X contains $(\pi \times \pi)^{-1}(\pi'_\alpha \times \pi'_\alpha)^{-1}(U)$ for some $\alpha \in A'$ and $u \in \mathcal{U}_\alpha$. Thus, p is an isomorphism of uniform spaces into X'. Note that here the fact that X is Hausdorff is needed to assure that π is injective.

Now assume that each X_α is complete and let $x' = (x_\alpha)_{\alpha\in A'}$, be a point of X'. Let $\alpha \in A, U \in \mathcal{U}_\alpha$ and choose $V \in \mathcal{U}_\alpha$ with $4V \subset U$. There is by (AIS) an $\alpha' \in A$ such that $\alpha < \alpha'$ and for $\alpha' < \beta < \gamma$

$$|p_{\alpha\gamma} - p_{\alpha\beta}p_{\beta\gamma}| < V. \tag{1}$$

For a fixed $\beta \in A'$ with $\alpha' < \beta$, there is $W \in \mathcal{U}_\beta$ such that

$$|x - y| < W \Rightarrow |p_{\alpha\beta}(x) - p_{\alpha\beta}(y)| < V. \tag{2}$$

Also, by (L), there is $\alpha'' > \beta$ such that for $\gamma \in A'$ with $\gamma > \alpha''$

$$|x_\beta - p_{\beta\gamma}(x_\gamma)| < W \tag{3}$$

and hence, by (2)

$$|p_{\alpha\beta}(x_\beta) - p_{\alpha\beta}p_{\beta\gamma}(x_\gamma)| < V. \tag{4}$$

Now, by (1) and (4), for $\gamma \in A'$ with $\gamma > \alpha''$,

$$|p_{\alpha\beta}(x_\beta) - p_{\alpha\gamma}(x_\gamma)| < 2V. \tag{5}$$

Hence for $\gamma, \delta \in A'$ with $\gamma, \delta > \alpha''$,

$$|p_{\alpha\gamma}(x_\gamma) - p_{\alpha\delta}(x_\delta)| < 4V \subset U. \tag{6}$$

This means that $\{p_{\alpha\gamma}(x_\gamma) : \gamma \in A'\}$ is a Cauchy net and, since X_α is complete and Hausdorff, it converges to a unique point x_α. Now it is seen that (2), (3) and (4) hold for any $\beta \in A$ with $\beta > \alpha'$. Also, there is $\gamma \in A'$ for which (5) holds as well as

$$|x_\alpha - p_{\alpha\gamma}(x_\gamma)| < V.$$

Hence $|x_\alpha - p_{\alpha\beta}(x_\beta)| < 3V \subset U$.

Thus $x_\alpha = \lim_{\beta\in A} p_{\alpha\beta}(x_\beta)$, $x = (x_\alpha)_{\alpha\in A}$ is a point of X with $\pi(x) = x'$, and π is surjective. □

Example 2. In Example 1, let X_i be a singleton for $i \geq 2$. Then the limit space of the AIS $((X_i, \mathcal{U}_i), p_{ij},\ i \geq 2)$ is a singleton.

Example 3. Let $A' = \{r_1, r_2, r_3, \dots\}$, where each r_i is an integer ≥ 2 chosen by induction on i so that

$$\big(1 + \frac{1}{2^{r_{i+1}}}\big)^3 < \big(1 + \frac{1}{2^{r_i}}\big).$$

Let $A = \{0\} \cup A'$ and for each $i \in A$ let $X_i = (1, (1 + \frac{1}{2^i})^2]$ carry the subspace uniformity inherited from R, the space of real numbers. For $i, j \in A'$ with $i < j$, we define $p_{ij} : X_j \to X_i$ by

$$p_{ij}(x) = \big(1 + \frac{1}{2^i}\big)\,\big(1 + \frac{1}{2^j}\big)x.$$

For $i \in A'$, we define $p_{oi} : X_i \to X_o$ by

$$p_{oi}(x) = \big(1 + \frac{1}{2^i}\big)x.$$

It is readily seen that (X_i, p_{ij}, A) constitutes an AIS with the limit $X = \emptyset$, for if $x_i \in X_i$, since $1 < p_{oi}(x_i) \leq (1 + \frac{1}{2^i})^3$, then $\lim_i p_{oi}(x_i) = 1 \notin X_o$. However, the subsystem (X_i, p_{ij}, A') contains at least one point, namely $(1 + \frac{1}{2^i})_{i\in A'}$.

3. Approximate inverse systems and dimension.

In this section, we show that the limit space X of an AIS made up of complete metric spaces $X_i, i \in N$, is in fact isomorphic with the inverse limit of a subset of these spaces. An immediate application is that $\dim X \leq n$ if $\dim X_i \leq n$ for each $i \in N$, and this result is generalized for an arbitrary AIS of complete metric spaces.

When dealing with several metric spaces, it is convenient to use the same symbol d for the metric on each one of them. Which metric is meant is usually abundantly clear from the context.

Proposition 8. *Let (X_i, p_{ij}, N) be an AIS of complete metric spaces with the limit space X. Then X is uniformly isomorphic with the limit space of an IS (X_i, π_{ij}, M), where M is a cofinal subset of N and $\pi_{ij} = p_{ij}$ whenever j is an immediate successor of i in M.*

PROOF: As in Section 2, $p_i : X \to X_i$ denotes canonical projection. If $M = \{m_1, m_2, \dots\}$ with $m_i < m_j$ for $i < j$, instead of $\pi_{m_i m_j}$ we will write q_{ij}. Note that q_{ij} is the composite of $p_{m_{j-1} m_j}, \dots, p_{m_i m_{i+1}}$. For each $i \in N$, we will choose $m_i \in N$ and $s_i \in R$ by induction on i so that

$$0 < s_{i+1} < s_i < 1/i, \tag{1}$$

$$m_i < m_{i+1}, \tag{2}$$

$$d(p_{m_i}(x), p_{m_i}(y)) < 3s_i \Rightarrow d(x,y) < 1/i \tag{3}$$

$$d(p_{m_i}, q_{ii+1} p_{m_{i+1}}) < \frac{1}{2} s_i, \tag{4}$$

$$i < j \text{ and } d(x,y) < s_j \Rightarrow d(q_{ij}(x), q_{ij}(y)) < 2^{-j} s_{j-1}, \tag{5}$$

$$\ell \leq m_i < m_{i+1} \leq m < n \Rightarrow d(p_{\ell n}, p_{\ell m} p_{mn}) < 2^{-i-2} s_i \tag{6}$$

and

$$m < n \leq m_i \text{ and } d(x,y) < s_i \Rightarrow d(p_{mn}(x), p_{mn}(y)) < s_{i-1}. \tag{7}$$

Assuming that s_i and m_i have been chosen for $i \leq j$ with the required properties, we can, using Proposition 2 and (AIS), first pick $m_{j+1} > m_j$ so that (4) and (6) with $i = j$ are satisfied. By Proposition 4, we can further assume that for some positive $t_{j+1} < s_j$,

$$d(p_{m_{j+1}}(x), p_{m_{j+1}}(y)) < 3t_{j+1} \Rightarrow d(x,y) < \frac{1}{j+1}.$$

Now using the uniform continuity of q_{ij+1}, $i \leq j$, and p_{mn}, $m < n \leq m_{j+1}$, we can find a positive $s_{j+1} < t_{j+1}$ that satisfies (3) and (7) with $i = j+1$ and (5) with j replaced by $j+1$. This completes the construction of m_i and s_i.

If $i < j$, it follows from (4) and (5) that

$$d(q_{ij} p_{m_j},\, q_{ij+1} p_{m_{j+1}}) < 2^{-j} s_{j-1}.$$

Hence, for $i < j < k$, by the triangle inequality,

$$d(q_{ij}p_{m_j},\, q_{ik}p_{m_k}) < s_{j-1}(\sum_{\ell=j}^{k-1} 2^{-\ell}) < \frac{1}{2}s_{j-1}\,. \tag{8}$$

Thus, for a fixed i, $\{q_{ij}p_{m_j}\}_{j=1}^{\infty}$ is a Cauchy sequence and, since X_{m_i} is complete, it converges to a uniformly continuous $\pi_i : X \to X_{m_i}$. Note that $q_{ij}\pi_j = \lim_k q_{ij}q_{jk}p_{m_k} = \lim_k q_{ik}p_{m_k} = \pi_i$. Hence, if Y is the limit space of the IS $\{X_{m_i}, q_{ij}, N\}$ and $q_i : Y \to X_{m_i}$ denotes the corresponding canonical projection, there is a uniformly continuous $\pi : X \to Y$ such that $q_i\pi = \pi_i$. It follows from (8) that for $i < j$,

$$d(q_{ij}p_{m_j}, \pi_i) \leq \frac{1}{2}s_{j-1}$$

and hence, by (4) and the triangle inequality

$$d(p_{m_i}, \pi_i) < s_i\,. \tag{9}$$

Now, using the triangle inequality and (9), if $d(\pi_i(x), \pi_i(y)) < s_i$, then $d(p_{m_i}(x), p_{m_i}(y)) < 3s_i$ and, by (3), $d(x,y) < \frac{1}{i}$. Thus the entourage $\{(x,y) : d(x,y) < \frac{1}{i}\}$ of X is refined by the inverse image under $\pi \times \pi$ of the entourage $\{(x,y) : d(q_i(x), q_i(y)) < s_i\}$ of Y. Hence π is a uniform isomorphism into Y.

To show that π is onto, let $y = (y_i) \in Y$. Then, using (6) and $y_i = q_{ii+1}(y_{i+1}) = p_{m_i m_{i+1}}(y_{i+1})$, for $\ell \leq m_{i-1}$,

$$d(p_{\ell m_{i+1}}(y_{i+1}),\, p_{\ell m_i}(y_i)) < 2^{-i-1}s_{i-1}\,.$$

Hence, if also $i < j$, using the triangle inequality,

$$d(p_{\ell m_i}(y_i),\, p_{\ell m_j}(y_j)) < 2^{-i}s_{i-1}\,. \tag{10}$$

Thus, $\{p_{\ell m_i}(y_i)\}_{i=1}^{\infty}$ is a Cauchy sequence for each $\ell \in N$ and so it converges to some point x_ℓ of X_ℓ. It follows from (10) that

$$d(x_\ell, p_{\ell m_i}(y_i)) \leq 2^{-i}s_{i-1}\,. \tag{11}$$

Let $m < m_{i-3} < m_{i-2} \leq n \leq m_{i-1}$. Then, using (11) and (7), (6), and (11), respectively, we obtain

$$\begin{aligned} d(p_{mn}(x_n), p_{mn}p_{nm_i}(y_i)) &< s_{i-2}\,, \\ d(p_{mm_i}(y_i), p_{mn}p_{nm_i}(y_i)) &< s_{i-3}\,, \\ d(x_m, p_{mm_i}(y_i)) &< s_{i-1}\,. \end{aligned}$$

It follows that $d(x_m, p_{mn}(x_n)) < 3s_{i-3}, x_m = \lim_n p_{mn}(x_n)$, and $x = (x_n) \in X$. Finally, by (11), $d(x_{m_j}, y_j) < s_j$ and, for $i < j$, by (5), $d(q_{ij}(x_{m_j}), y_i) < s_{j-1}$. This implies that $y_i = \lim_j q_{ij}p_{m_j}(x)$, so that $y = \pi(x)$, and hence π is onto. □

Corollary 3. *If X is the limit space of an AIS (X_i, p_{ij}, N) of complete metric spaces with $\dim X_i \leq n$ for each $i \in N$, then $\dim X \leq n$.*

PROOF: With the notation of Proposition 8, $\dim X = \dim Y \leq n$ according to the inverse limit theorem of Nagami [10]. □

A generalization of Corollary 3 is possible in terms of the dimension function Dim introduced by the author in [1]. A subset of a uniform space X is called uniformly open if it is of the form $f^{-1}(G)$, where G is an open set of a metric space Y and $f : X \to Y$ is uniformly continuous. $\operatorname{Dim} X \leq n$ iff every finite uniformly open cover of X has a finite uniformly open refinement of order $\leq n$. Clearly, Dim and dim agree on metric spaces.

We first need the following result which is interesting in itself.

Proposition 9. *Let X be the limit space of an AIS $(X_\alpha, p_{\alpha\beta}, A)$ of metric spaces. For each $i \in N$, let $f_i : X \to M_i$ be a uniformly continuous function into a metric space M_i. Then there is an AIS $(X_\alpha, p_{\alpha\beta}, B)$ over a countable subset B of A with limit space Y, and uniformly continuous $\pi : X \to Y$ and $g_i : \pi(X) \to M_i$ with $f_i = g_i\pi$, $i \in N$.*

PROOF: Let $p_\alpha : X \to X_\alpha$ denote the canonical projection and $f = \prod_{i\in N} f_i : X \to \prod_{i\in N} M_i$. Using (AIS) and Propositions 2 and 4, we can readily pick by induction an increasing sequence $B = \{\alpha_i\}$ in A and a sequence $\{s_i\}$ of positive reals such that for $i < j < k$,

$$d(p_{\alpha_i\alpha_k}, p_{\alpha_i\alpha_j}p_{\alpha_j\alpha_k}) < \frac{1}{j}\,, \tag{1}$$

$$d(p_{\alpha_i}, p_{\alpha_i\alpha_j}p_{\alpha_j}) < \frac{1}{j} \tag{2}$$

and

$$d(p_{\alpha_i}(x), p_{\alpha_i}(y)) < s_i \Rightarrow d(f(x), f(y)) < \frac{1}{i}\,. \tag{3}$$

Now, (1) assures that $(X_\alpha, p_{\alpha\beta}, B)$ is an AIS, and (2), by Proposition 6, that there is a uniformly continuous $\pi : X \to Y$ with $q_i\pi = p_{\alpha_i}$, where $q_i : Y \to X_{\alpha_i}$ denotes the canonical projection. Now, define $g(y) = f(x)$ if $y = \pi(x)$. Then g is well-defined by (3), and $g\pi = f$, so that, if g_i denotes g followed by the projection $\prod_{i\in N} M_i \to M_i$, then $f_i = g_i\pi$. Finally, (3) with $q_i\pi = p_{\alpha_i}$ assure that g and hence each g_i is uniformly continuous. □

Proposition 10. *Let X be the limit space of an AIS $(X_\alpha, p_{\alpha\beta}, A)$ of complete metric spaces with $\operatorname{Dim} X_\alpha \leq n$. Then $\operatorname{Dim} X \leq n$ (cf. [2, Theorem]).*

PROOF: Let $\{H_j : j \in J\}$ be a finite uniformly open cover of X. Let $f_j : X \to M_j$ be a uniformly continuous function into a metric space M_j with $H_j = f_j^{-1}(G_j)$ for some open set G_j of M_j. With the notation of Proposition 9, $\dim \pi(X) \leq \dim Y \leq n$ by Corollary 3. Now, $\{g_j^{-1}(G_j) : j \in J\}$ is a finite open cover of $\pi(X)$ and so it has an open refinement $\{U_j : j \in J\}$ of order $\leq n$. Then $\{\pi^{-1}(U_j) : j \in J\}$ is a uniformly open refinement of $\{H_j : j \in J\}$ of order $\leq n$. Hence $\operatorname{Dim} X \leq n$. □

4. Approximate inverse systems of polyhedra.

Our purpose in this section is to generalize the main result of Mardešić and Rubin in [9]. Every topologically complete topological space X with $\dim X \leq n$ is shown to be the limit space of an AIS of metric polyhedra of dim $\leq n$. If X is strongly paracompact, the polyhedra can be taken to be locally finite. Note that because $\dim X \leq \operatorname{Dim} X$ for X completely paracompact [2] and every finite-dimensional metric polyhedron is complete [7], the results of this section have a converse in Proposition 10.

The reference below to a metric polyhedron (P, d) entails that P is the realization $|K|$ of some fixed simplicial complex K and d is a metric on P uniformly equivalent to the standard metric induced by K as in [14]. Note that for finite-dimensional polyhedra the metric considered in [7] is uniformly equivalent to the standard metric. It is convenient to call a mapping $f : X \to P$ locally finite if there is an open cover of X consisting of sets G with $f(G)$ covered by a finite number of simplexes of K. If P is locally finite, more precisely, if K is locally finite, then the star of a vertex $v, \operatorname{st}(v)$, meets only a finite number of simplexes, and hence every continuous $f : X \to P$ is locally finite. The lemmas that follow generalize the results of [8].

Lemma 1. *Let (P, d) be a finite-dimensional metric polyhedron, $f : X \to P$ locally finite and $\varepsilon > 0$. Then there is a subpolyhedron Q of P and a locally finite surjection $g : X \to Q$ with $d(f, g) \leq \varepsilon$.*

Proof: Let K be a subdivision of the original triangulation of P with d-mesh of every simplex $\leq \varepsilon$. Let $\{s_\alpha : \alpha < \omega\}$ be a bijective enumeration of all simplexes of K. By transfinite induction, we define a map π_α into P for each $\alpha < \omega$ as follows. If s_α is the union of points of the form $\pi_{\alpha_n} \pi_{\alpha_{n-1}} \dots \pi_{\alpha_1} f(x)$ with $\alpha_1 < \alpha_2 \dots < \alpha_n < \alpha$, then we let π_α be the identity on P. If some point p_α of s_α is not of the above form, we let π_α on $s - \{p_\alpha\}$ be the projection from p_α into the boundary of s if s is a simplex containing p_α, and $\pi_{\alpha/s} =$ identity if $p_\alpha \notin s$. Let $\pi : f(X) \to P$ be the composite of all π_α, $\alpha < \omega$, with π_α applied before π_β for $\alpha < \beta$, and $g = \pi f$. Note that on each simplex, g is the composite of f with only a finite number of π_α's and the local finiteness of f assures that g is continuous and hence locally finite. It is readily checked that $g(X) = Q$ is a polyhedron and $d(g, f) \leq \varepsilon$. □

Lemma 2. *Let X be a Tychonoff space with $\dim X \leq n, f : X \to M$ a map into a metric space and $\mathcal{V}$ an open cover of M. Then there is a locally finite cozero refinement $\mathcal{U}$ of the cover $f^{-1}(\mathcal{V})$ of X with order $\leq n$.*

Proof: By Pasynkov's factorization theorem [11], there is a metric space L and continuous $g : X \to L$ and $h : L \to M$ with $\dim L \leq n$ and $f = hg$. Next, the open cover $h^{-1}(\mathcal{V})$ of L has a locally finite open refinement $\mathcal{W}$ of order $\leq n$, and we need only set $\mathcal{U} = g^{-1}(\mathcal{W})$. □

If $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a locally finite cozero cover of X, there is a canonical mapping $f : X \to |\mathcal{N}(\mathcal{U})|$, where $\mathcal{N}(\mathcal{U})$ denotes the nerve of $\mathcal{U}$, defined as follows. Let $f_\alpha : X \to I$ be continuous with $f_\alpha^{-1}(0, 1] = U_\alpha$ and $\sum_{\alpha \in A} f_\alpha = 1$. Then $f(x)$ is the point with α-th barycentric coordinate $f_\alpha(x)$. Note that f is then a locally finite mapping.

Lemma 3. *Let X be a Tychonoff space with $\dim X \leq n$, (P,d) a metric finite-dimensional polyhedron, $f : X \to P$ a locally finite mapping and $\varepsilon > 0$. Then there is a subpolyhedron Q of P with $\dim Q \leq n$ and a locally finite surjection $g : X \to Q$ with $d(f,g) \leq \varepsilon$.*

Proof: Let K be a subdivision of the original triangulation of P with d-mesh of each simplex $\leq \frac{\varepsilon}{2}$. Let $\mathcal{V}$ be the set of all vertices of K and $\mathcal{V}^*$ the collection of all their stars. By Lemma 2, there exists a locally finite cozero cover $\mathcal{U}$ of X of order $\leq n$ refining $f^{-1}(\mathcal{V}^*)$. Let $\phi : \mathcal{U} \to \mathcal{V}$ be a function such that $U \subset f^{-1}(\mathrm{st}(\phi(U)))$. Let $h_1 : X \to |\mathcal{N}(\mathcal{U})|$ be a canonical mapping and $h_2 : |\mathcal{N}(\mathcal{U})| \to P$ the simplicial mapping sending U to $\phi(U)$. It is readily seen that $h = h_2 h_1$ is a locally finite mapping into P with $h(x)$ lying in the open simplex of K containing $f(x)$ and hence $d(f,h) \leq \frac{\varepsilon}{2}$. Now, by Lemma 1, there is a subpolyhedron Q of $h_2(|\mathcal{N}(\mathcal{U})|)$, necessarily of dim $\leq n$, and a locally finite surjection $g : X \to Q$ with $d(h,g) \leq \frac{\varepsilon}{2}$ and hence $d(f,g) \leq \varepsilon$. □

Lemma 4. *Let X be a Tychonoff space with $\dim X \leq n$, (P_i, d_i), $i = 1, \dots, k$, finite-dimensional metric polyhedra, $f_i : X \to P_i$ locally finite maps and $\varepsilon > 0$. Then there exists a metric polyhedron (Q,d) with $\dim Q \leq n$, a locally finite surjection $g : X \to Q$ and uniformly continuous $p_i : Q \to P_i$ with $d_i(f_i, p_i g) \leq \varepsilon$.*

Proof: $P = \prod_{i=1}^k P_i$ with the metric $d = \prod_{i=1}^k d_i$ is a finite-dimensional polyhedron [7, p. 60], and $f = \prod_{i=1}^k f_i : X \to P$ is a locally finite map. Lemma 3 now provides a locally finite surjection $g : X \to Q$ onto a subpolyhedron Q of P with $\dim Q \leq n$ and $d(f,g) \leq \varepsilon$. Let p_i be the restriction of the canonical projection $P \to P_i$ to Q. □

Proposition 11. *A topologically complete Hausdorff space X with $\dim X \leq n$ is homeomorphic with the limit space of an AIS $((P_\alpha, d_\alpha), p_{\alpha\beta}, A)$ of metric polyhedra (P_α, d_α) with $\dim P_\alpha \leq n$ and surjective uniformly continuous maps $p_{\alpha\beta}$.*

Proof: Start with the complete uniformity $\mathcal{V}$ on X of weight w. In view of Lemma 2, there is a collection $\{\mathcal{U}_\lambda : \lambda \in \Lambda\}$, where $|\Lambda| = w$, of locally finite cozero covers of X of order $\leq n$ such that each uniform cover of $\mathcal{V}$ is refined by some $\mathcal{U}_\lambda$. Let $P_\lambda = |\mathcal{N}(\mathcal{U}_\lambda)|$ with its standard metric and $f_\lambda : X \to P_\lambda$ a canonical mapping.

Let the set A of all finite non-void subsets of Λ be ordered by inclusion. For $\alpha = \{\lambda\}$, we let $P_\alpha = P_\lambda$ and $f_\alpha = f_\lambda$. For $\alpha \in A$, we construct by induction on $|\alpha|$, using the fact that each α has only a finite number of predecessors in conjunction with Lemma 4 and the definition of uniform continuity, a metric polyhedron (P_α, d_α) with $\dim P_\alpha \leq n$, a locally finite surjection $f_\alpha : X \to P_\alpha$, a uniformly continuous $p_{\alpha\beta} : P_\beta \to P_\alpha$ for $\alpha < \beta$, and a positive ε_α less than 1 such that

$$d_\alpha(p_{\alpha\beta} f_\beta, f_\alpha) < 2^{|\alpha| - |\beta|} \varepsilon_\alpha \tag{1}$$

and

$$d_\beta(x,y) < \varepsilon_\beta \Rightarrow d(p_{\alpha\beta}(x), p_{\alpha\beta}(y)) < 2^{|\alpha| - |\beta|}. \tag{2}$$

In view of (1), given $\alpha < \beta < \gamma$,

$$d(p_{\beta\gamma} f_\gamma, f_\beta) < \varepsilon_\beta \,.$$

Then, by (2), for any point $y = f_\gamma(x)$ of P_γ,

$$d(p_{\alpha\beta} p_{\beta\gamma}(y), p_{\alpha\beta} f_\beta(x)) < 2^{|\alpha|-|\beta|}. \tag{3}$$

Hence, by (1) and (3), and (1), respectively,

$$d(p_{\alpha\beta} p_{\beta\gamma}(y), f_\alpha(x)) < 2^{|\alpha|-|\beta|+1} \tag{4}$$

and

$$d(p_{\alpha\gamma}(y), f_\alpha(x)) < 2^{|\alpha|-|\gamma|}. \tag{5}$$

It now follows from (4) and (5) that

$$d(p_{\alpha\gamma}, p_{\alpha\beta} p_{\beta\gamma}) < 2^{|\alpha|-|\beta|+2}.$$

Thus $((P_\alpha, d_\alpha), p_{\alpha\beta}, A)$ is an AIS. Let Y be its limit space and $p_\alpha : Y \to P_\alpha$ the canonical projection. Let X be endowed with the smallest uniformity $\mathcal{U}$ that makes each f_α uniformly continuous. Note that the weight $(\mathcal{U}) \le w$, $\mathcal{U}$ is finer than $\mathcal{V}$ and hence $\mathcal{U}$ is complete. In view of (1) and Proposition 6, there is a uniformly continuous $g : X \to Y$ with $p_\alpha g = f_\alpha$ for each $\alpha \in A$. If x_1, x_2 are distinct points of X, for some $\alpha = \{\lambda\} \in A$, $f_\alpha(x_1) \ne f_\alpha(x_2)$. Hence, $g(x_1) \ne g(x_2)$ and g is injective. Now since g and each p_α are uniformly continuous, then $g : X \to g(X)$ is an isomorphism by the definition of $\mathcal{U}$. Hence, being complete, $g(X)$ is a closed subspace of Y. Finally, if $y \in Y - g(X)$, then y and $g(X)$ are distant, so that, by Proposition 4, for some $\alpha \in A$, $p_\alpha(y)$ and $p_\alpha g(X) = f_\alpha(X)$ are distant in P_α, which contradicts the fact that f_α is surjective. Thus, $Y = g(X)$ is isomorphic with X, and this completes the proof. □

An immediate corollary is Nagami's theorem 27-7 in [10].

Corollary 4. *If X is completely metrizable with $\dim X \le n$, then X is the limit space of an inverse sequence of metric polyhedra with $\dim \le n$.*

Proof: In the proof of Proposition 11, A can be taken to be countable, and the result then follows from Proposition 8. □

The following result is readily established by making trivial modifications to the proof of Proposition 11. What is needed is the fact that an open cover of a strongly paracompact space X with $\dim X \le n$ has a star-finite cozero shrinking of order $\le n$.

Proposition 12. *A strongly paracompact Hausdorff space X with $\dim X \leq n$ is homeomorphic with the limit space of an AIS $((P_\alpha, d_\alpha), p_{\alpha\beta}, A)$ of metric locally finite polyhedra (P_α, d_α) with $\dim P_\alpha \leq n$ and surjective uniformly continuous maps $p_{\alpha\beta}$.*

The following result generalizes Freudenthal's original result for compact metric spaces [5]. It follows from Propositions 8 and 12 in the same manner as Corollary 4 follows from Propositions 8 and 11.

Corollary 5. *A completely metrizable separable space X with $\dim X \leq n$ is the limit space of an inverse sequence of metric separable locally finite polyhedra with $\dim \leq n$.*

We conclude this section with a proof of a version of the factorization theorem that we use in Section 5, using results of this paper. The reader will have noticed that in Lemma 2 for a compact space X, the use of the factorization theorem was superfluous.

Proposition 13. *Let X be a Tychonoff space with $\dim X \leq n$ and $f : X \to M$ a continuous function onto a compact metric space. Then there is a compact metric space L with $\dim L \leq n$ and continuous $g : X \to L$ and $h : L \to M$ with $hg = f$ (cf. [8] and [12]).*

PROOF: Since the Stone–Čech compactification of X has the same dim as X, we can assume that X is compact. Let $(P_\alpha, p_{\alpha\beta}, A)$ be the AIS of polyhedra with $\dim P_\alpha \leq n$ obtained in Proposition 11. With the notation of the proof of that proposition, we can assume that f is uniformly continuous w.r.t. $\mathcal{V}$ and hence $\mathcal{U}$. Then, by Proposition 9, there is a countable subset B of A such that $(P_\alpha, p_{\alpha\beta}, B)$ is an AIS with limit Y, and uniformly continuous $g : X \to Y$ and $h : L = g(X) \to M$ with $f = hg$. Finally, by Proposition 10, $\dim Y \leq n$ and hence $\dim L \leq n$. $\square$

5. An application of inverse limits.

The purpose of this section is to give a proof of the equality $\dim X = \operatorname{Ind} X$ for metric spaces, using essentially only results on inverse limits. This proof has apparently not been previously noticed.

A mapping $f : X \to Y$ between metric spaces is called special if for each $j \in N$, there is an open cover $\{V_i^j : i \in N\}$ of $f(X)$ such that each $f^{-1}(V_i^j)$ is the union of a discrete collection $\{V_{i\alpha}^j : \alpha \in A\}$ of open sets of X with mesh $\leq \frac{1}{j}$. Note that if f is special, the same is true of its restriction to a subspace of X.

Proposition 14. *If $f : X \to Y$ is special, then $\operatorname{Ind} X \leq \operatorname{Ind} Y$.*

PROOF: The proof is by induction on $n = \operatorname{Ind} Y$, the cases $n = -1$ or ∞ being trivial. Let $\operatorname{Ind} Y = n \geq 0$ and suppose the result holds provided the range has $\operatorname{Ind} \leq n-1$. By the subset theorem, we can suppose that f is onto. Let V_i^j and $V_{i\alpha}^j$ be as in the definition of a special mapping. Then there is an open refinement $\{U_i^j : i \in N\}$ of the open cover $\{V_i^j : i \in N\}$ of Y with $\overline{U_i^j} \subset V_i^j$ and $\operatorname{Ind} \partial(U_i^j) \leq$

$n-1$, where ∂ denotes boundary. Put $U^j_{i\alpha} = V^j_{i\alpha} \cap f^{-1}(U^j_i)$. By the induction hypothesis, $\operatorname{Ind} f^{-1}(\partial(U^j_i)) \leq n-1$, and hence $\operatorname{Ind} \partial(U^j_{i\alpha}) \leq n-1$ by the subset theorem. Let E, F be disjoint closed sets of X and, for $i, j \in N$, set

$$G^j_i = \bigcup(U^j_{i\alpha} : \overline{U^j_{i\alpha}} \cap F = \emptyset)$$

and

$$H^j_i = \bigcup(U^j_{i\alpha} : \overline{U^j_{i\alpha}} \cap E = \emptyset).$$

The following facts are readily checked.

(1) $$\operatorname{Ind} \partial(G^j_i) \leq n-1, \quad \operatorname{Ind} \partial(H^j_i) \leq n-1,$$

(2) $$X = \bigcup_{i,j \in N} G^j_i \cup H^j_i,$$

(3) $$\overline{G^j_i} \cap F = \emptyset \text{ and } \overline{H^j_i} \cap E = \emptyset.$$

Now by a well-known technique [4, Lemma 2.3.16], using (2) and (3), one can construct a partition L between E and F which is a closed subspace of $\bigcup_{i,j\in N} \partial(G^j_i) \cup \partial(H^j_i)$. Then by (1) and the countable sum and the subset theorems, $\operatorname{Ind} L \leq n-1$, and hence $\operatorname{Ind} X \leq n$. □

Lemma 5. *For a compact metric space X, $\operatorname{Ind} X \leq \dim X$.*

Proof: The proof is by induction on $n = \dim X$, for $n = -1$ or ∞ the proof being trivial. Assume then that $\dim X = n \geq 0$ and that the result holds for all compact spaces with $\dim \leq n-1$. By Proposition 12, X is the limit space of an inverse sequence (P_i, p_{ij}, N), where each P_i is a finite n-dimensional polyhedron. Let $p_i : X \to P_i$ denote the canonical projection. Let V be an open neighbourhodd of a closed subset E of X. By the compactness of X, there is $k \in N$ and an open neighbourhood U of $p_k(E)$ with $E \subset W = p_k^{-1}(U) \subset \overline{W} \subset V$. Then $(\partial p_{ki}^{-1}(U), p_{ij}, k \leq i < j)$ is an inverse sequence with limit as a compact metric subspace Y of X containing ∂W. It is well known that the boundary of a subset of R^n has $\dim \leq n-1$ (see Theorem 1.8.10 and the problem 1.8.D of [4], and Theorem IV.3 of [6]). It follows that $\dim \partial p_{ki}^{-1}(U) \leq n-1$ and by Corollary 3 and the subset theorem, $\dim \partial W \leq \dim Y \leq n-1$. Finally, by the induction hypothesis, $\operatorname{Ind} \partial W \leq n-1$ and hence $\operatorname{Ind} X \leq n$. □

A similar argument to that employed above was originally employed in [8] and [11].

Proposition 15. *For a metric space X, $\dim X = \operatorname{Ind} X$.*

Proof: It suffices to show that $\dim X \leq n < \infty$ implies $\operatorname{Ind} X \leq n$ since $\dim \leq \operatorname{Ind}$ holds for all normal spaces. For each $j \in N$, let $\{U^j_{i\alpha} : i \in N, \alpha \in A\}$ be a σ-discrete open cover of X of mesh $\leq \frac{1}{j}$. Let $f^j_i : X \to I$ be continuous with

$(f_i^j)^{-1}(0,1] = U_i^j = \bigcup(U_{i\alpha}^j : \alpha \in A)$. Then $f = \prod_{i,j\in N} f_i^j : X \to I^N$ is a special mapping. By Proposition 13, there is a compact metric space Y with $\dim Y \le n$ and continuous $g : X \to Y$ and $h : Y \to I^N$ with $hg = f$. This last property implies that g is a special mapping. Finally, by Proposition 14 and Lemma 15, $\operatorname{Ind} X \le \operatorname{Ind} Y \le \dim Y \le n$. $\square$

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