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## Fixed points of asymptotically regular mappings in spaces with uniformly normal structure

JAROSŁAW GÓRNICKI

*Abstract.* It is proved that: for every Banach space  $X$  which has uniformly normal structure there exists a  $k > 1$  with the property: if  $A$  is a nonempty bounded closed convex subset of  $X$  and  $T : A \rightarrow A$  is an asymptotically regular mapping such that

$$\liminf_{n \rightarrow \infty} \|T^n\| < k,$$

where  $\|T\|$  is the Lipschitz constant (norm) of  $T$ , then  $T$  has a fixed point in  $A$ .

*Keywords:* asymptotically regular mappings, uniformly normal structure, fixed points

*Classification:* 47H10

### 1. Introduction.

The concept of uniformly normal structure is due to A.A. Gillespie and B.B. Williams [7]. A Banach space  $X$  has uniformly normal structure if

$$N(X) = \sup\{r_A(A) : A \subset X, \text{ convex, diam } A = 1\} < 1,$$

where

$$r_A(A) = \inf \{ \sup\{\|x - y\| : y \in A\} : x \in A \}.$$

It was proved in [4], [2] that  $N(X) \leq 1 - \delta_X(1)$ ; thus  $\varepsilon_0(X) < 1$  implies uniformly normal structure. In the paper [11] X.T. Yu proved that if  $X$  is a uniformly smooth space (or more generally,  $\lim_{t \downarrow 0} \rho_X(t)t^{-1} < \frac{1}{2}$ ), then  $X$  has a uniformly normal structure. Also, in [12] it was proved that uniformly normal structure does not necessarily imply that the space has good geometric properties.

The concept of asymptotic regularity is due to F. Browder and V. Petryshyn [1]. A mapping  $T : X \rightarrow X$  is said to be asymptotically regular if

$$\lim_{n \rightarrow \infty} \|T^{n+1}x - T^n x\| = 0$$

for all  $x \in X$ .

If  $T$  is nonexpansive, then  $T_\lambda := \lambda \cdot I + (1 - \lambda) \cdot T$  is asymptotically regular for all  $0 < \lambda < 1$  (see [6]).

Recently P.K. Lin in [10] has constructed a uniformly asymptotically regular Lipschitzian mapping acting on a weakly compact subset of  $l^2$  which has no fixed point.

E.A. Lifshitz (see [5]) associated with each metric space  $(M, d)$  a constant  $\kappa(M) \geq 1$ . Define Lifshitz characteristic  $\kappa_0(X)$  to be the infimum of  $\kappa(C)$  where  $C$  ranges over all nonempty closed bounded convex subsets of the Banach space  $X$ . D.J. Downing and B. Turett [5] proved the following

**Theorem 1.** *Let  $X$  be a Banach space.*

- (1) *Then  $\varepsilon_0(X) < 1$  if and only if  $\kappa_0(X) > 1$ .*
- (2) *If  $\gamma > 1$  satisfies  $\gamma(1 - \delta_X(\gamma^{-1})) = 1$ , then  $\gamma \leq \kappa_0(X)$ .*

In [8] the present author proved the following

**Theorem 2.** *Let  $X$  be a Banach space with the Lifshitz characteristic  $\kappa_0(X) > 1$  and let  $C$  be a nonempty bounded closed convex subset of  $X$ . If  $T : C \rightarrow C$  is an asymptotically regular mapping such that*

$$\liminf_{n \rightarrow \infty} \|T^n\| < \kappa_0(X),$$

*then  $T$  has a fixed point in  $C$ .*

**2. Main result.**

The main result of this paper is interesting in the Banach spaces  $X$  which satisfy the conditions:  $\varepsilon_0(X) \geq 1$  and  $N(X) < 1$  (cf. [3]).

We start with the following

**Lemma 1** [3]. *Let  $X$  be a Banach space with  $N(X) < 1$ . Then for every bounded sequence  $\{x_n\}$  there exists a point  $z \in \overline{\text{conv}}\{x_n\}$ , such that:*

- (i)  $\limsup_{n \rightarrow \infty} \|z - x_n\| \leq N(X) \cdot \limsup_{s \rightarrow \infty} \{\|x_n - x_m\| : n, m \geq s\}$ ,
- (ii) *for every  $y \in X$ ,  $\|z - y\| \leq \limsup_{n \rightarrow \infty} \|y - x_n\|$ .*

**Lemma 2** [9]. *Let  $A$  be a nonempty closed convex subset of a Banach space  $X$  and let  $\{n_i\}$  be an increasing sequence of natural numbers. Assume that  $T : A \rightarrow A$  is an asymptotically regular mapping such that for some  $m \in \mathbb{N}$ ,  $T^m$  is continuous. If*

$$\hat{r}(x) = \limsup_{i \rightarrow \infty} \|x - T^{n_i}u\| = 0$$

*for some  $u \in A$  and  $x \in A$ , then  $Tx = x$ .*

**Theorem 3.** *Let  $A$  be a nonempty bounded closed convex subset of a Banach space  $X$  which has uniformly normal structure, i.e.  $N(X) < 1$ . If  $T : A \rightarrow A$  is an asymptotically regular mapping such that*

$$\liminf_{n \rightarrow \infty} \|T^n\| = k < [N(X)]^{-1/2},$$

then  $T$  has a fixed point in  $A$ .

PROOF: Let  $T : A \rightarrow A$  and let  $\{n_i\}$  be a sequence of natural numbers such that

$$\liminf_{n \rightarrow \infty} \|T^n\| = \lim_{i \rightarrow \infty} \|T^{n_i}\| = k < [N(X)]^{-1/2}.$$

Consider the sequence  $\{T^{n_i}x\}$  for an  $x \in A$ . Let  $z(x)$  be a point satisfying Lemma 1 for  $\{T^{n_i}x\}$ . Let  $r(x) = \limsup_{i \rightarrow \infty} \|T^{n_i}x - x\|$ . By the condition (i) of Lemma 1, we

have

$$\begin{aligned} (1) \quad & \limsup_{i \rightarrow \infty} \|T^{n_i}x - z\| \leq N(X) \cdot \lim_{s \rightarrow \infty} \sup\{\|T^{n_i}x - T^{n_j}x\| : n_i, n_j \geq s\} \leq \\ & \leq N(X) \cdot \limsup_{i \rightarrow \infty} (\limsup_{j \rightarrow \infty} \|T^{n_i}x - T^{n_j}x\|) \leq \\ & \leq N(X) \cdot \limsup_{i \rightarrow \infty} (\limsup_{j \rightarrow \infty} (\|T^{n_i}x - T^{n_i+n_j}x\| + \|T^{n_i+n_j}x - T^{n_j}x\|)) \leq \\ & \leq N(X) \cdot \limsup_{i \rightarrow \infty} (\limsup_{j \rightarrow \infty} (\|T^{n_i}\| \cdot \|x - T^{n_j}x\| + \sum_{v=0}^{n_i-1} \|T^{n_j+v+1}x - T^{n_j+v}x\|)) \leq \\ & \leq N(X) \cdot \limsup_{i \rightarrow \infty} \|T^{n_i}\| \cdot \limsup_{j \rightarrow \infty} \|x - T^{n_j}x\| = \\ & = k \cdot N(X) \cdot \limsup_{j \rightarrow \infty} \|x - T^{n_j}x\|. \end{aligned}$$

Moreover, for  $i > 1$ , we have

$$\begin{aligned} & \|T^{n_i}z - z\| \leq \limsup_{j \rightarrow \infty} \|T^{n_i}z - T^{n_j}z\| \leq \\ & \leq \limsup_{j \rightarrow \infty} (\|T^{n_i}z - T^{n_i+n_j}z\| + \|T^{n_i+n_j}z - T^{n_j}z\|) \leq \\ (2) \quad & \leq \limsup_{j \rightarrow \infty} (\|T^{n_i}\| \cdot \|z - T^{n_j}z\| + \sum_{v=0}^{n_i-1} \|T^{n_j+v+1}z - T^{n_j+v}z\|) \leq \\ & \leq \|T^{n_i}\| \cdot \limsup_{j \rightarrow \infty} \|z - T^{n_j}z\|. \end{aligned}$$

By (1) and (2)

$$(3) \quad r(z) \leq k^2 \cdot N(X) \cdot r(x) = a \cdot r(x), \quad \text{with } a < 1.$$

Define a sequence  $\{x_m\}$  in the following way:  $x_1$  is an arbitrarily chosen point of  $A$ ,  $x_{m+1} = z(x_m)$ . Then  $\{x_m\}$  is a Cauchy sequence. In fact, we have

$$\begin{aligned} \|x_{m+1} - x_m\| & \leq \|x_{m+1} - T^{n_i}x_m\| + \|T^{n_i}x_m - x_m\| \leq \\ & \leq \|x_{m+1} - T^{n_i}x_m\| + r(x_m). \end{aligned}$$

Taking the limit superior as  $i \rightarrow +\infty$ ,

$$\begin{aligned} \|x_{m+1} - x_m\| &\leq \limsup_{i \rightarrow \infty} \|x_{m+1} - T^{n_i}x_m\| + r(x_m) \leq \\ &\leq k \cdot N(X) \cdot r(x_m) + r(x_m) = [1 + k \cdot N(X)] \cdot r(x_m). \end{aligned}$$

Hence, by (3)

$$\|x_{m+1} - x_m\| \leq [1 + k \cdot N(X)] \cdot r(x_m) \leq [1 + k \cdot N(X)] \cdot a^m \cdot r(x_1) \rightarrow 0$$

as  $m \rightarrow +\infty$ . Let  $x_0 = \lim_{m \rightarrow \infty} x_m$ . Finally

$$\begin{aligned} \|x_0 - T^{n_i}x_0\| &\leq \|x_0 - x_m\| + \|x_m - T^{n_i}x_m\| + \|T^{n_i}x_m - T^{n_i}x_0\| \leq \\ &\leq (1 + \|T^{n_i}\|) \cdot \|x_0 - x_m\| + \|x_m - T^{n_i}x_m\|. \end{aligned}$$

Taking the limit superior as  $i \rightarrow +\infty$  on both sides we get

$$\begin{aligned} \limsup_{i \rightarrow \infty} \|x_0 - T^{n_i}x_0\| &\leq (1 + k) \cdot \|x_0 - x_m\| + r(x_m) \leq \\ &\leq (1 + k) \cdot \|x_0 - x_m\| + a^m \cdot r(x_1) \rightarrow 0 \end{aligned}$$

as  $m \rightarrow +\infty$ . Therefore, by Lemma 2,  $Tx_0 = x_0$ . □

For James spaces  $X_M = (l^2, |\cdot|_M)$ , where  $|\cdot|_M = \max\{\|\cdot\|_2, M \cdot \|\cdot\|_\infty\}$ , ( $M \geq 1$ ) we have

1)

$$\varepsilon_0(X_M) = \begin{cases} 2 \cdot (M^2 - 1)^{1/2} & \text{for } M < \sqrt{2}, \\ 2 & \text{for } M > \sqrt{2}, \end{cases}$$

and  $\varepsilon_0(X_M) < 1$  if and only if  $M < \frac{\sqrt{5}}{2}$ ;

2) for  $1 \leq M < \frac{\sqrt{5}}{2}$ , the condition  $\gamma < [N(X_M)]^{-1/2}$  is weaker than  $\gamma < \gamma_0$ , where  $\gamma_0$  is the unique solution of  $x(1 - \delta_{X_M}(\frac{1}{x})) = 1$ ;

and

$$N(X_M) = \frac{M}{\sqrt{2}} \text{ for } 1 \leq M \leq \sqrt{2}, [3].$$

Combining these results we get the following

**Corollary 1.** *Let  $A$  be a nonempty bounded closed convex subset of a James space  $X_M$ ,  $1 \leq M < \sqrt{2}$ . If  $T : A \rightarrow A$  is an asymptotically regular mapping such that*

$$\liminf_{n \rightarrow \infty} \|T^n\| < \frac{2^{1/4}}{\sqrt{M}},$$

*then  $T$  has a fixed point in  $A$ .*

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