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Smoothness for systems of degenerate variational inequalities with natural growth

Martin Fuchs

Abstract. We extend a regularity theorem of Hildebrandt and Widman [3] to certain degenerate systems of variational inequalities and prove Hölder-continuity of solutions which are in some sense stationary.

Keywords: variational inequalities, regularity theory

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0. Introduction.
We consider systems of variational inequalities of the form

$$\int_{\Omega} A(u)|Du|^{p-2}Du \cdot D(v-u) \, dx \geq \int_{\Omega} f(\cdot,u,Du) \cdot (v-u) \, dx$$

for all $v \in K := H^{1,p}(\Omega,K)$ such that $\text{spt}(u-v) \subset \subset \Omega$, where $K$ is a convex set in $\mathbb{R}^N$ and $p$ denotes some real number in the interval $[2,n]$, $n$ denoting the dimension of the domain $\Omega$. Our main purpose is to prove (partial) regularity for solutions $u \in K$ of (0.1) in the case that the right-hand side is of natural growth, i.e. we require

$$|f(x,y,Q)| \leq a \cdot (|Q|^p + 1)$$

for some positive constant $a$. To my knowledge there is only a theorem of Hildebrandt and Widman [3] concerning the quadratic case $p = 2$ which can be summarized as follows:

$$\text{If } A \geq \lambda > 0 \text{ and if } a < \lambda / \text{diam } K$$

is satisfied then any solution $u$ of (0.1) is of class $C^{0,\alpha}$ on the whole domain $\Omega$.

Since these authors make use of the Green’s function technique it is rather clear that for general $p > 2$ one has to find completely new arguments. We start with the observation that (0.2) is sufficient to prove a Caccioppoli inequality for $u$ giving $Du \in L^q_{\text{loc}}$ for some $q > p$ and hence partial regularity apart from a closed singular set of vanishing $\mathcal{H}^{n-q}$-measure. Of course the convexity of $K$ is essential in two ways: it is needed to derive Caccioppoli’s inequality and to show that local solutions $w$ of $D(|Dw|^{p-2}Dw) = 0$ for boundary values $u$ are admissible. Unfortunately we did not succeed in proving everywhere regularity by the way giving
a complete extension of the above mentioned theorem of Hildebrandt and Widman. Our contribution concerns the following case: suppose that \( f \) is of the special form \( f(x, y, Q) = \frac{1}{2} DA(y) |Q|^p \) and that in addition \( u \) is a stationary point of the functional \( F(u) := \int_{\Omega} A(u) |Du|^p \, dx \) with respect to reparametrizations of \( \Omega \). This enables us to consider blow-up sequences at possible singularities which are shown to converge strongly to a homogeneous (degree zero) tangent map \( u_0 \) in the space \( H^{1,p}_{\text{loc}}(\Omega) \) and from (0.2) it follows that \( u_0 \) must be trivial so that the singular set is empty. Hence our main result can be summarized as follows:

Suppose that \( u \in \mathbb{K} \) satisfies

\[
\lim_{t \to 0} \frac{d}{dt} \left[ F(u + t(v - u), B) - F(u, B) \right] \geq 0 \quad \text{for all } v \in \mathbb{K} \quad \text{such that } \text{spt}(u - v) \subset \subset \Omega.
\]

Then if (0.2) holds and if \( u \) is also stationary we have \( u \in C^{0,\alpha}(\Omega) \).

1. Notations and results.

We here specify our assumptions and introduce some notations which will be used throughout the paper. Let \( B_r(x_0) := \{ x \in \mathbb{R}^n : |x - x_0| < r \} \), we often write \( B_r \) when \( x_0 \) is fixed and use the symbol \( B \) to denote the open unit ball with center at 0. For a compact convex set \( K \) in \( \mathbb{R}^N \) and a real number \( 2 \leq p < n \) we introduce the class \( \mathbb{K} := \{ u \in H^{1,p}(B, \mathbb{R}^N) : u(x) \in K \ \text{a.e.} \} \) of all vector-valued Sobolev functions with values in the prescribed set \( K \). Moreover, we are given a smooth function \( A: \mathbb{R}^n \to \mathbb{R} \) with the property

\[
\lambda \leq A(y), \quad y \in K,
\]

for some positive number \( \lambda \). For the functions \( u \in \mathbb{K} \) and balls \( B_r(x_0) \subset B \) we then define the energy

\[
F(u, B_r(x_0)) := \int_{B_r(x_0)} A(u) |Du|^p \, dx.
\]

**Theorem 1.1.** Suppose \( u \in \mathbb{K} \) satisfies

\[
\lim_{t \to 0} t^{-1} \cdot \left[ F(u + t(v - u), B) - F(u, B) \right] \geq 0
\]

for all \( v \in \mathbb{K} \) with the property \( \text{spt}(u - v) \subset \subset B \). Then, if the smallness condition

\[
\sup_K |DA| < 2 \cdot \lambda \cdot (\text{diam } K)^{-1}
\]

holds, we have \( u \in C^{0,\alpha}(B') \) for some open subset \( B' \) of \( B \) such that \( \mathcal{H}^{n-p}(B - B') = 0 \).

**Definition.** A function \( u \in \mathbb{K} \) is a stationary point of \( F(\cdot, B) \) iff

\[
\frac{d}{dt} F(u_t, B) = 0, \quad u_t(x) := u(x + t \cdot X(x)),
\]

holds for all vectorfields \( X \in C^1_0(B, \mathbb{R}^n) \).
Theorem 1.2. Let \( u \in K \) denote a stationary point of \( \mathcal{F}(\cdot, B) \) which in addition satisfies (1.2). Then \( u \in C^{0,\alpha}(B) \) provided the smallness condition (1.3) is satisfied.

Remarks: 1) Theorems 1.1, 1.2 easily extend to functionals of the form

\[
u \rightarrow \int_B A(u) (a_{\alpha\beta} D_\alpha u \cdot D_\beta u)^{p/2} dx
\]

with elliptic coefficients \( a_{\alpha\beta} : B \to \mathbb{R} \).

2) We conjecture that (1.2), (1.3) are sufficient to prove everywhere regularity.

3) Under suitable smallness conditions relating \( \lambda, \text{diam}(K) \) and the growth constant \( a \) in

\[
|f(x, y, Q)| \leq a(|Q|^p + 1),
\]

a partial regularity result in the spirit of Theorem 1.1 can be deduced for solutions \( u \in K \) of the variational inequality

\[
\int_B A(u) |Du|^{p-2} Du \cdot (Dv - Du) dx \geq \int_B f(\cdot, u, Du) \cdot (v - u) dx, \quad v \in K, \text{spt}(u - v) \subset B,
\]

but again we are unable to exclude singular points.

2. Proof of the partial regularity Theorem 1.1.

Clearly inequality (1.2) is equivalent to

\[
\int_B A(u) |Du|^{p-2} Du \cdot (Dv - Du) dx \geq \int_B f(\cdot, u, Du) \cdot (v - u) dx, \quad v \in K, \text{spt}(u - v) \subset B,
\]

for all \( v \in K \) such that \( \text{spt}(u - v) \subset B \). Consider a ball \( B_{2R}(x_0) \subset B \) and a cut-off function

\[
\eta \in C^1_0(B_{2R}(x_0), [0, 1]), \quad \eta = 1 \text{ on } B_R(x_0), \quad |D\eta| \leq 2 \cdot R^{-1}.
\]

Then

\[
v := u + \eta^p(u_{2R} - u), \quad u_{2R} := \int_{B_{2R}(x_0)} u dx,
\]

is admissible in (2.1) and a standard calculation using (1.3) implies Caccioppoli’s inequality

\[
\int_{B_R(x_0)} |Du|^p dx \leq c_1 \cdot R^{-p} \int_{B_{2R}(x_0)} |u - u_{2R}|^p dx
\]

for some absolute constant \( c_1 \) independent of \( u \) and the ball \( B_R(x_0) \). Quoting [G] we find an exponent \( q > p \) such that

\[
Du \in L^q_{\text{loc}}(B, \mathbb{R}^N)
\]
and the following reverse Hölder inequality holds

\[(2.3) \quad \left( \int_{B_R(x_0)} |Du|^q \, dx \right)^{1/q} \leq c_3 \left( \int_{B_{2R}(x_0)} |Du|^p \, dx \right)^{1/p}.
\]

Let \( w \in H^{1,p}(B_R(x_0), \mathbb{R}^N) \) denote the unique minimizer of the functional

\[ F_0(v) := A(u_R) \cdot \int_{B_R(x_0)} |Dv|^p \, dx \]

for boundary values \( u \mid_{\partial B_R(x_0)} \). Since \( u(B_R(x_0)) \subset K \) and since \( K \) is convex, one easily checks (for example by projecting \( v \) onto the set \( K \)) that \( v \) respects the side condition and therefore is admissible in (2.1) provided we integrate over the ball \( B_R(x_0) \). As in [1, Lemma 3.3] we then can prove the following comparison inequality

\[(2.4) \quad \int_{B_R(x_0)} |Du - Dv|^p \, dx \leq c_4 \cdot \left[ R^{p-n} \int_{B_R(x_0)} |Du|^p \, dx \right]^{1-p/q} \int_{B_{2R}(x_0)} |Du|^p \, dx.
\]

Note that the proof of (2.4) combines (2.3) with standard ellipticity estimates. On the other hand we know from [5] that

\[ \int_{B_\rho(x_0)} |Dv|^p \, dx \leq c_r \left( \frac{\rho}{R} \right)^n \int_{B_R(x_0)} |Dv|^p \, dx, \quad 0 < \rho \leq R,
\]

which gives on account of (2.4):

**Lemma 2.1.** Suppose that \( u \in K \) satisfies (1.2) and that the smallness condition (1.3) holds. Then there exist constants \( \varepsilon, \alpha \in (0,1) \) (independent of \( u \)) with the following property: If

\[(2.5) \quad R^{p-n} \int_{B_R(x_0)} |Du|^p \, dx < \varepsilon
\]

holds for some ball \( B_R(x_0) \subset B \) then \( u \in C^{0,\alpha}(B_{R/2}(x_0)) \) and

\[ |u(x) - u(y)| \leq c \cdot |x - y|^\alpha, \quad x, y \in B_{R/2}(x_0),
\]

with \( 0 < c < \infty \) independent of \( u \).

This proves Theorem 1.1 and in view of Caccioppoli’s inequality (2.2) we see that a point \( x_0 \in B \) is a regular point if and only if

\[(2.5)' \quad \int_{B_R(x_0)} |u - u_R|^p \, dx < \varepsilon'
\]

holds for some ball \( B_R(x_0) \subset B \) and a suitable small constant \( \varepsilon' \in (0,1) \).
3. Monotonicity and everywhere regularity.

The following lemma is essentially due to Price \cite{4} (for $p = 2$).

**Lemma 3.1.** Let $u \in \mathcal{K}$ satisfy (1.4). Then we have

$$0 = \int_B A(u) |Du|^{p-2} \left[ |Du|^2 \text{div} X - pD\alpha u \cdot D\beta u X^\beta \right] dx$$

for all vectorfields $X \in C^1_0(B, \mathbb{R}^n)$.

□

By applying (3.1) to fields of the form $X(x) = \gamma(|x|) x$ for a function $\gamma \in C^1(\mathbb{R})$ such that $(0 < \rho < 1)$

$$\gamma' \leq 0, \quad \gamma = 1 \quad \text{on} \quad (-\infty, \rho/2], \quad \gamma = 0 \quad \text{on} \quad (\rho, \infty),$$

we get

**Lemma 3.2** (Monotonicity formula). Suppose that $u \in \mathcal{K}$ satisfies (1.4). Then

$$R^{p-n} \int_{B_R} A(u)|Du|^p \, dx - r^{p-n} \int_{B_r} A(u)|Du|^p \, dx$$

$$= p \cdot \int_{B_R-B_r} A(u)|Du|^{p-2} \cdot |Du|^2 \cdot |x|^{p-n} \, dx$$

holds for balls $B_r(0) \subset B_R(0) \subset B$.

**Remarks:**
1) $D_r u$ denotes the radial derivative: $D_r u^i(x) := \nabla u^i(x) \cdot \frac{x}{|x|}$.
2) A similar formula is valid for balls with center $x_0 \in B$.

We now come to the proof of Theorem 1.2: Let all the assumptions of Theorem 1.2 hold; it clearly suffices to show

$$\lim_{R \uparrow 0} R^{p-n} \int_{B_R(0)} |Du|^p \, dx = 0,$$

i.e. $0 \in \text{Reg}(u)$ (= the regular set of $u$). To this purpose define a sequence $r_k \downarrow 0$ and consider the scaled maps $u_k(z) := u(r_k z)$, $z \in B$, which belong to the class $\mathcal{K}$ and satisfy (2.1) for all $v \in \mathcal{K}$, $\text{spt}(u_k - v) \subset B$. Since

$$\sup_k \|u_k\|_{H^{1,p}(B)} < \infty,$$

we may extract a subsequence (again denoted by $u_k$) such that

$$u_k \to: u_0 \quad \text{in} \quad L^p_{\text{loc}}, \quad u_k \rightharpoonup u_0 \quad \text{weakly in} \quad H^{1,p}_{\text{loc}}$$
and pointwise a.e. The limit \( u_0 \) is in the class \( \mathbb{K} \) and let us suppose for the moment that we already know

\[ u_k \to u_0 \quad \text{strongly in } \mathcal{H}^{1,p}_{\text{loc}}. \]  

(3.3)

We then fix an arbitrary point \( \xi \in K \) and a function \( \eta \in C_0^1(0,1) \), \( 0 \leq \eta \leq 1 \), and apply (2.1) with \( u \) replaced by \( u_k \) and \( v(x) := u_k(x) + \eta(|x|) (\xi - u_k(x)) \). (\( v \) is admissible since \( \text{Im } v \subset K \) and \( \text{spt}(u_k - v) \subset \subset B \).) On account of (3.3) we may pass to the limit \( k \to \infty \) in order to deduce

\[
\int_B A(u_0) D u_0 \cdot D (\eta(u_0 - \xi)) |D u|^{p-2} \, dx \leq \int_B \frac{1}{2} D A u_0 \cdot \eta(|\xi - u_0|) |D u_0|^p \, dx,
\]

which gives (recall (1.3))

\[ \delta \int_B \eta \cdot |D u_0|^p \, dx + \int_B A(u_0) |D u_0|^{p-2} D_\alpha u_0 \cdot (u_0 - \xi) \eta'(x) x_\alpha \cdot |x|^{-1} \, dx \leq 0 \]

(3.4)

for some \( \delta > 0 \). By scaling (3.1) is valid also for \( u_k \) and strong convergence \( u_k \to u_0 \) in \( \mathcal{H}^{1,p}_{\text{loc}} \) shows that (3.1) holds for the limit \( u_0 \). Thus Lemma 3.2 extends to \( u_0 \).

Applying Lemma 3.2 to \( u \) we see that \( \Phi(t) := t^{p-n} \int_{B_t} A(u) |Du|^p \, dx \) is an increasing function so that \( L := \lim_{t \downarrow 0} \Phi(t) \) exists. On the other hand we have for any \( 0 < R < 1 \)

\[
R^{p-n} \int_{B_R} A(u_0) |Du_0|^p \, dx = \lim_{k \to \infty} R^{p-n} \int_{B_R} A(u_k) |Du_k|^p \, dx = \lim_{k \to \infty} (r_k \cdot R)^{p-n} \int_{B_{r_k \cdot R}} A(u) |Du|^p \, dx = L,
\]

which shows \( D_r u_0 \equiv 0 \). Inserting this result into (3.4) we finally arrive at

\[
\int_B \eta \cdot |D u_0|^p \, dx = 0
\]

so that \( Du_0 = 0 \) a.e. on \( B \), and in conclusion

\[
0 = R^{p-n} \int_{B_R(0)} |Du_0|^p \, dx = \lim_{k \to \infty} R^{p-n} \int_{B_R(0)} |Du_k|^p \, dx
\]

\[
= \lim_{k \to \infty} (r_k \cdot R)^{p-n} \int_{B_{r_k \cdot R}(0)} |Du|^p \, dx,
\]
which proves (3.2).

It remains to verify (3.3): Choose a point \( x \in B \) such that

\[
\int_{B_r(x)} |u_0 - (u_0)_r|^p \, dz < \varepsilon'
\]

holds for some ball \( B_r(x) \subset B \) with \( \varepsilon' \) being defined in (2.5). For \( k \) sufficiently large we then have

\[
\int_{B_r(x)} |u_k - (u_k)_r|^p \, dz < \varepsilon'
\]

and since Lemma 2.1 applies to \( u_k \) we get the apriori estimate

\[
[u_k]_{C^{0,\alpha}(B_{r/2}(x))} \leq c \leq \infty
\]

for the Hölder-seminorms with \( c \) independent of \( k \). Arzela’s theorem implies \( u_k \to u_0 \) uniformly on \( B_{r/2}(x) \), especially \( u_0 \in C^{0,\alpha}(B_{r/2}(x)) \).

Let \( S_0 \) denote the interior singular set of \( u_0 \). The preceding arguments show

\[
S_0 \subset \Sigma_0 := \{ x \in B : \liminf_{r \downarrow 0} \int_{B_r(x)} |u_0 - (u_0)_r|^p \, dz > 0 \},
\]

so that \( \mathcal{H}^{n-p}(S_0) \leq \mathcal{H}^{n-p}(\Sigma_0) = 0 \). Fix a number \( t \in (0, 1) \) and some small \( \delta > 0 \) and choose a covering

\[
\Sigma_0 \cap B_t \subset \bigcup_{i=1}^{\infty} B_i, \quad B_i := B_{r_i}(x_i) \subset B,
\]

with the property \( \sum_{i=1}^{\infty} r_i^{n-p} < \delta \). Then we have the following estimate for the energies on the set \( 0 =: \bigcup_{i=1}^{\infty} B_i \):

\[
\int_O |Du_k|^p \, dx \leq \sum_{i=1}^{\infty} \int_{B_i} |Du_k|^p \, dx
\]

\[
\leq (\text{monotonicity formula for } u_k) \leq c \cdot \sum_{i=1}^{\infty} r_i^{n-p} \int_B |Du_k|^p \, dx
\]

\[
= c \cdot \sum_{i=1}^{\infty} r_i^{n-p} (r_k^{p-n} \int_{B_{r_k}} |Du|^p \, dx)
\]

\[
\leq (\text{monotonicity formula}) \leq c' \cdot \delta \cdot \int_B |Du|^p \, dx.
\]
In order to control the energies on the remaining part we choose \( \eta \in C_0^1(\bar{B}, [0, 1]) \) such that \( \eta \equiv 1 \) on \( \tilde{B}_\ell - O \) and \( \text{spt} \, \eta \cap S_0 = \emptyset \). For \( k \in \mathbb{N} \) we have

\[
\int_B A(u_k) \, |D(u_k)|^{p-2} D(u_k) \cdot D(u_k - v) \, dx \\
\leq \int_B \frac{1}{2} \, DA(u_k) \cdot (v - u_k) \, |Du_k|^p \, dx, \\
v \in \mathbb{K}, \, \text{spt} \, (u_k - v) \subset B;
\]

choosing \( v := u_k + \eta^p \cdot (u_\ell - u_k) \) in (3.5)_k and \( v := u_\ell + \eta^p(u_k - u_\ell) \) in (3.5)_\ell we arrive at

\[
\int_B \left( A(u_k) \, D(u_k) \cdot D(u_k - u_\ell) \, |Du_k|^{p-2} \right. \\
\left. - A(u_\ell) \, D(u_\ell) \cdot D(u_k - u_\ell) \, |Du_\ell|^{p-2} \right) \cdot \eta^p \, dx \\
\leq c_1 \cdot \int_B |D\eta^p| \cdot |u_k - u_\ell| \cdot \{|Du_\ell|^{p-1} + |Du_k|^{p-1}\} \, dx \\
+ c_2 \cdot \int_B \eta^p \cdot |u_k - u_\ell| \cdot \{|Du_\ell|^p + |Du_k|^p\} \, dx,
\]

which turns into an estimate of the form \((\tau > 0 \text{ a positive constant})\)

\[
\tau \cdot \int_B \eta^p \cdot |Du_k - Du_\ell|^p \, dx \\
\leq c_3 \cdot \int_B |u_k - u_\ell| \cdot \left( |D\eta^p| \cdot \{|Du_\ell|^{p-1} + |Du_k|^{p-1}\} \\
+ \eta^p \cdot \{|Du_k|^p + |Du_\ell|^p\} \right) \, dx.
\]

Recalling \( \sup \{|u_\ell(x) - u_k(x)| : x \in \text{spt} \, \eta\} \xrightarrow{\ell,k \to \infty} 0 \) we see

\[
\int_B \eta^p |Du_\ell - Du_k|^p \, dx \xrightarrow{\ell,k \to \infty} 0
\]

so that \( \{Du_k\} \) is a Cauchy-sequence in \( L^p_{\text{loc}}(B) \) which completes the proof of (3.3). \( \square \)

References


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