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A general upper bound in extremal theory of sequences

MARTIN KLazar

Abstract. We investigate the extremal function \(f(u, n)\) which, for a given finite sequence \(u\) over \(k\) symbols, is defined as the maximum length \(m\) of a sequence \(v = a_1a_2\ldots a_m\) of integers such that 1) \(1 \leq a_i \leq n\), 2) \(a_i = a_j, i \neq j\) implies \(|i - j| \geq k\) and 3) \(v\) contains no subsequence of the type \(u\). We prove that \(f(u, n)\) is very near to be linear in \(n\) for any fixed \(u\) of length greater than 4, namely that

\[f(u, n) = O(n^{2^O(\alpha(n)|u|^{-4})}).\]

Here \(|u|\) is the length of \(u\) and \(\alpha(n)\) is the inverse to the Ackermann function and goes to infinity very slowly. This result extends the estimates in [S] and [ASS] which treat the case \(u = abababa\ldots\) and is achieved by similar methods.

Keywords: sequence, Davenport-Schinzel sequence, length, upper bound

Classification: 05D99

INTRODUCTION

In the Extremal theory of sequences we investigate the quantity

\[f(u, n) = \max\{|v| \mid u \not\leq v, \|v\| \leq n, v \text{ is } \|u\|-regular\}.

Here \(u\) and \(v\) are finite sequences of arbitrary symbols, \(n\) is a nonnegative integer, \(|v|\) stands for the length of \(v\) and \(\|v\|\) denotes the cardinality of \(S(v)\), the set of all symbols that occur in \(v\). If there is a subsequence \(s\) in \(v\) such that \(s\) differs from \(u\) only in the names of the symbols we write \(u \not\leq v\) and say that \(v\) contains \(u\). For instance \(v_1 = 123245131\) contains both \(u_1 = xxyy\) and \(u_2 = ababa\). A sequence \(u = a_1a_2\ldots a_m\) is called \(k\)-regular if \(a_i = a_j, i \neq j\) implies \(|i - j| \geq k\). Example: \(v_1\) and \(u_2\) are 2-regular but are not 3-regular and \(u_1\) is not 2-regular. If \(u = a_1a_2\ldots a_m\) and \(a_i = a \in S(u)\) then we shall refer to \(a_i\) as to the \(a\)-letter.

The function \(f(u, n)\) extends in a natural way the function \(F = f(ababa, n)\) investigated at first by Davenport and Schinzel in [DS]. They proved the upper bound \(F = O(n \log n / \log \log n)\) that was later improved by Szemerédi to \(O(n \log^* n)\) ([Sz]). Here \(\log^* n\) is the minimum number of iterations of the power function \(2^m\) (starting with \(m = 1\)) which are needed to get a number greater or equal to \(n\). The question whether \(F = O(n)\) \((f(abab, n) = 2n - 1\) trivially) remained open until 1986 when it was answered by Hart and Sharir in [HS] negatively. They showed

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that $F = \Theta(n\alpha(n))$ where $\alpha(n)$ goes to infinity but very slowly (a precise definition of $\alpha(n)$ will be given in the second part of this paper). M. Sharir obtained later

$$f(al_s, n) = O(n\alpha(n)O(\alpha(n)^{s-5}))$$

for arbitrary alternating sequence $al_s = abab\ldots$ of the length $s \geq 5$ ([S]). Recently almost tight estimates were derived ([ASS]):

$$f(al_s, n) \leq n2^{(\alpha(n))^s-\frac{s-5}{2}}\log_2\alpha(n)+C_s(n)$$

for $s \geq 5$ odd

$$f(al_s, n) \leq n2^{(\alpha(n))^s-\frac{s-4}{2}}+C_s(n)$$

for $s \geq 6$ even

$$f(al_s, n) = \Omega(n2^{K_s.(\alpha(n))^s-\frac{s-4}{2}}+Q_s(n))$$

for $s \geq 6$ even

where $K_s = \frac{1}{(\frac{s-1}{2})!}$ and $C_s(n)$ and $Q_s(n)$ are asymptotically smaller than the main terms. For $s = 6$ even, $f(ababab, n) = \Theta(n2^{\alpha(n)})$ ([ASS]). How complex the previous formulae may seem on the first view, one thing is clear: $f(al_s, n)$ is almost almost linear in $n$ for all $s$.

The first aim of this paper is to show that the same is true for arbitrary sequence $u$. The second aim is to give a brief and clear idea about the techniques developed by Agarwal, Hart, Sharir and Shor for obtaining almost linear upper bounds on $f(al_s, n)$ to the reader that is not familiar with them.

In the first part we show a simple method that leads to the upper bound $f(u, n) = O(n^2)$ for all $u$. Then, in the second part, we use a slightly generalized method of [S] to derive the estimate

$$f(u, n) = O(n.3^{\alpha(n)^{|u|}-4})).$$

**PART 1**

We first define a modification of the function $f(u, n)$ for $l$-regular sequences:

$$f(u, n, l) = \max\{|v| \mid u \not\subseteq v, \|v\| \leq n, v \text{ is } l\text{-regular}\}$$

where $l \geq \|u\|$. 

**Lemma 1.1.** a) $f(u, n, l)$ is defined and finite for any $n \geq 1$ and moreover $f(u, n, l) = O(|u|.|u|.|n|.|u|)$.

b) $f(u, n, l) \leq f(u, n, k) \leq (1+f(u, l-1, k))f(u, n, l)$ for all $l > k \geq \|u\|, n \geq 1$.

**Proof:** ad a) We suppose there is at least one repetition in $u$, otherwise the function $f(u, n, l)$ is constant. If $n < l$ then $f(u, n, l) = n$. If $n \geq l$ then any $l$-regular sequence $v$ satisfying $|v| \geq \|u\|.(\frac{n}{\|u\|})(\|u\|-1)+1$ must contain $u$. We split $v = v_1v_2\ldots v_c w$ so that $|v_i| = |v_k| = \|u\|$ and $c = (\|u\|-1)(\frac{n}{\|u\|})+1$. According to the Dirichlet Principle there exist $|u|$ indices $1 \leq i_1 < i_2 < \ldots < i_{\|u\|} \leq c$ that $S(v_{i_1}) = S(v_{i_2}) = \ldots = S(v_{i_{\|u\|}})$. Thus $u \leq v_{i_1}v_{i_2}\ldots v_{i_{\|u\|}}$. 


ad b) The first inequality is obvious. Suppose \( v = a_1a_2\ldots a_m \) is \( k \)-regular, does not contain \( u \) and \( \|v\| \leq n \). We choose a subsequence \( v^* \) of \( v \) in this way: we start with \( v^* = a_1 \) and \( i = 1 \) and search for the minimum \( j \) such that \( j > i \) and \( v^*a_j \) is \( l \)-regular. If such a \( j \) exists then we put \( v^* = v^*a_j \) and \( i = j \) and repeat. Otherwise the algorithm terminates. Obviously \( \|v^*\| \leq n \) and \( v^* \) is \( l \)-regular. Moreover \( |v| \leq (1 + f(u,l-1,k))|v^*| \) because any interval \( I \) in \( v \) omitted by the previous algorithm satisfies \( |I| \leq l - 1 \). We got the second inequality. \( \square \)

**Definition 1.2.** Let \( u, v \) be sequences. We write \( u \leq v \) if \( u \leq v^* \) for all \( v^* \) obtained from \( v \) by restricting \( v \) to some \( \|u\| \) symbols. Thus in this case \( v \) contains \( u \) in all possible ways.

**Lemma 1.3.** For any sequence \( u \) there exist positive integers \( m \) and \( s \) such that \( u \leq v \) whenever \( \|v\| \geq m \) and \( al_s \leq v \).

Before proving this lemma we derive the main result of this section.

**Theorem 1.4.** \( f(u,n) = O(n^2) \) for all sequences \( u \). The constant in \( O \) depends on \( u \).

**Proof:** Let \( m = m(u) \) be as in Lemma 1.3. According to Lemma 1.1b) we have
\[
 f(u,n) = f(u,n,\|u\|) \leq (1 + f(u,m-1,\|u\|))f(u,n,m).
\]
Suppose \( v \) is \( m \)-regular, \( \|v\| \leq n, u \not\leq v \) and \( |v| = f(u,n,m) \). It suffices to estimate the number \( c \) in the splitting \( v = v_1v_2\ldots vcw \) where \( |v_i| = \|v_i\| = m \) and \( |w| \leq m-1 \). Let \( s = s(u) \) stand for the second number of Lemma 1.3. For any \( v_i \) there exist symbols \( a, b \in S(v_i) \) such that \( v \) restricted on the symbols \( \{a, b\} \) does not contain \( al_s \). Otherwise \( u \leq v \) according to Lemma 1.3. But the mapping
\[
 F : \{v_1, v_2, \ldots, v_c\} \to \binom{S(v)}{2}
\]
that maps any \( v_i \) on a pair \( \{a, b\} \) mentioned above maps only at most \( s - 2 \) \( v_i \)'s on one pair because of the property of the symbols \( \{a, b\} \). Thus \( c \leq (s - 2)\binom{n}{2} \). Finally
\[
f(u,n) \leq (1 + f(u,m-1,\|u\|))m(c+1) \leq (1 + f(u,m-1,\|u\|))m(1 + (s - 2)\binom{n}{2}).
\]
Thus
\[
f(u,n) = O(n^2).
\]
\( \square \)

It remains to prove Lemma 1.3. We use the following well known:

**Lemma 1.5** (Erdős P., Szekeres G. 1935 [ES]). Any \((n-1)^2 + 1\)-term sequence (of integers) contains a \( n \)-term monotone subsequence.

**Proof of Lemma 1.3:** We denote by \( X(k,l) \) the set of all sequences of the form \( y_1y_2\ldots y_l \) where \( y_i = x_1x_2\ldots x_k \) or \( y_i = x_k\ldots x_{k-1}x_1 \) for \( k \) distinct symbols \( x_1, x_2, \ldots, x_k \). Thus \( |X(k,l)| = 2^l \) and \( |u| = kl \) and \( \|u\| = k \) for any \( u \in X(k,l) \). Since \( u \leq w \) for any \( w \in X(\|u\|, |u|) \), it suffices to prove the following claim.
Claim. For all positive integers $k$ and $l$ there exist positive integers $m$ and $s$ such that $w \leq v$ for some $w \in X(k, l)$ whenever $\|v\| \geq m$ and $al_s \leq \leq v$.

Proof of the claim: We put $s = 2l$ and $m = k_1$ where $k_1 = k$ and $k_{t-1} = 4(t-1)k_1^2 + 3$ for $t = l, l-1, \ldots, 2$. Suppose $v$ meets the prescribed conditions. We prove by induction that for all $t = 1, 2, \ldots, l$ there exists $w \in X(k_{t-1}, t)$ such that $w \leq v$. For $t = 1$ this is obvious. Suppose it is true for $t-1 \geq 1$. We have $w \in X(k_{t-1}, t-1), w \leq v$. We take a fixed $w$-copy $U$ in $v$ and split $v$ into $t - 1$ intervals $v = v_1v_2 \ldots v_{t-1}$ where $v_i$ contains $i$-th part of $U$ (i.e. $y_i$). $U$ consists of $k_{t-1}(t-1)$ letters $x_i^j$, $j = 1 \ldots t - 1$, $i = 1 \ldots k_{t-1}$ in $v$, $x_i^j$ occur in $v_j$, $a < b$ implies that $x_a^j$ precedes $x_b^j$ and $x_1^1 = x_1^q, x_2^1 = x_2^q, \ldots$ or $x_1^p = x_{k_{t-1}}^q, x_2^p = x_{k_{t-1}}^q, \ldots$ for all $p, q$. It remains to give names to the symbols — say that $x_i^j$ is $z(i)$-letter for $i = 1, 2, \ldots, k_{t-1}$. There must be other $z(i)$-letters in $v$ besides those in $U (al_s \leq \leq v)$. We use the pairs of symbols $(z(1), z(k_{t-1})), (z(2), z(k_{t-1} - 1)), \ldots, (z(L), z(k_{t-1} - L + 1)), L = [k_{t-1}/2] - 1$. The Dirichlet Principle implies that there are at least $|M| \geq L$ and an index $r \in \{1, 2, \ldots, t-1\}$ such that $z(i)z(k_{t-1} - i + 1)z(i)$ or $z(k_{t-1} - i + 1)z(i)z(k_{t-1} - i + 1)$ is a 3-term subsequence of $v_r$ for any $i \in M$. We used that $al_s \leq \leq v$ and $s > 2(t - 1)$. We can suppose w.l.o.g. $r = 1$. Thus we have 2-term subsequence $z(k_{t-1} - i + 1)z(i)$ of $v_1$ for any $i \in M$ (the opposite order than in $U$). The $z(L+1)$-letter $x_1^{L+1}$ (lies in $U$) splits $v_1$ on two intervals $v_1 = v''_1$. There are at least $|M|/2$ $i$’s in $M$ such that $z(i)$-letter occurs in $v''_1$ or there are $|M|/2$ $i$’s in $M$ such that $z(k_{t-1} - i + 1)$-letter occurs in $v'_1$. We obtained $t$ separated areas — namely $v'_1, v''_1, v_2, \ldots, v_{t-1}$ — in which $z(i)$-letter occurs for at least $|M|/2$ $i$’s. From those at least $|M|/2$ $i$’s we choose according to Lemma 1.5 at least $\sqrt{|M|}/2$ $i$’s in such a way that we obtain a $w$-copy in $v$, $w' \in X([\sqrt{|M|}/2], t)$. We are finished because $[\sqrt{|M|}/2] \geq [\sqrt{L/2(t-1)}] \geq \ldots \geq k_t$.

Remark 1.6. If we estimate $k_{t-1} = 4(t-1)k_1^2 + 3 \leq t(2k_1)^2$ then it may be easily derived that it suffices to put in Lemma 1.3 $s = 2|u|, m = (4|u|, \|u\|)^2|u|-1$.

Part 2

In this section we prove a result far stronger than $f(u, n) = O(n^2)$. At first we give the precise (standard) definition of $\alpha(n)$.

For any function $B : \mathbb{N} \rightarrow \mathbb{N}$ the symbol $B^{(s)}(n)$ denotes $B(B(\ldots(B(n))\ldots))$ ($s$ times). We define further the functional inverse of $B$ as $B^{-1}(n) = \min\{s \geq 1 \mid B(s) \geq n\}$. For nondecreasing and unbounded $B$ the functional inverse $B^{-1}$ is nondecreasing and unbounded as well. The functions $A_k(n)$ are defined by induction:

$$A_k(1) = 2, A_1(n) = 2n$$ and $A_k(n) = A_{k-1}(n)$.

Thus $A_2(n) = 2^n, A_3(n) = 2^{2^n}$, $n$ times. The Ackermann function is diagonal function of that schema: $A(n) = A_n(n)$. The function $\alpha(n)$ is defined as $\alpha(n) =$
A general upper bound in extremal theory of sequences

A\(^{-1}(n)\). Apart the hierarchy \(A_1, A_2, \ldots\) \((A_{i+1} \text{ grows to infinity much faster than } A_i\)) we have the hierarchy \(a_1, a_2, \ldots, a_i = A^{-1}_i\) \((a_{i+1} \text{ grows to infinity much more slowly than } a_i)\). Thus \(a_1(n) = \lfloor \frac{n}{2} \rfloor\), \(a_2(n) = \lceil \log_2 n \rceil\), \(a_3(n) = \log^*(n)\), \ldots\). The function \(a\) is far “lazier” than any \(a_i\). It is easy to prove for \(a_i\) a recurrent formula
\[
\alpha_{i+1}(a_i(m)) = \alpha_{i+1}(m) - 1 \quad \text{for all } i \geq 1, m \geq 3.
\]

Further \([\text{ASS}]\)
\[
(1) \quad \alpha_{\alpha(n)+1}(n) \leq 4 \quad \text{for all } n \geq 1.
\]

A sequence \(u\) is called a 1-chain if no symbol occurs repeatedly in \(u\). \(Y(k, l)\) denotes the set of all sequences of the form \(xy_1y_2\ldots y_l\) where any \(y_i\) is a permutation of \(k\) fixed symbols \(x_1, x_2, \ldots, x_k\). \(Y(k, l) \not\subseteq v\) means that \(u \leq v\) for no \(u \in Y(k, l)\). We modify a bit the function \(\Psi_s(m, n)\) of [S] and introduce the function
\[
\Psi^s_r(m, n) = \max\{|v| : \text{\(v\) is \(r\)-regular, \(\|v\| \leq n\), \(v = v_1v_2\ldots v_m\ where any \(v_i\) is 1-chain and } Y(r, s) \not\subseteq v\}\}.
\]

We will estimate \(f(u, n)\) in four steps. We will proceed induction on \(s\). At first we estimate \(\Psi^s_r(m, n)\). Then we derive, supposing we have an upper bound on \(\Psi^{s-1}_r(m, n)\), a recurrent inequality for \(\Psi^s_r(m, n)\). In the third step using that inequality the upper bound considered in Step 2 is extended on \(\Psi^s_r(m, n)\). Finally we estimate \(f(u, n)\) by appropriate \(\Psi^s_r(m, n)\).

**Step 1.**

**Lemma 2.1.** \(\Psi^2_r(m, n) \leq 2rn\).

**Proof:** Suppose \(v\) is \(r\)-regular, \(\|v\| \leq n\) and \(Y(r, 3) \not\subseteq v\) (we ignore here the first variable in \(\Psi\)). We split \(v = v_1v_2\ldots v_ew\) where \(|v_i| = \|v_i\| = r\) and \(|w| < r\). Any \(v_i\) must contain the first letter or the last letter of some symbol (otherwise \(u \leq v\) for some \(u \in Y(r, 3)\)). Thus
\[
|v| = cr + |w| \leq (2\|v\| - |w|)r + |w| \leq 2rn.
\]

□

**Step 2.**

**Lemma 2.2.** Suppose \(\Psi^{s-1}_r(m, n) \leq F_{s-1}(m)m + G_{s-1}(m)n\) for \(m, n \geq 1\) for some nondecreasing functions \(F_{s-1}, G_{s-1} : \mathbb{N} \to \mathbb{N}\). Then for any partition \(m = m_1 + \ldots + m_b, m_i \geq 1, 1 < b < m\) there exists a partition \(n = n_0 + n_1 + \ldots + n_b, n_i \geq 0\) such that
\[
(3) \quad \Psi^s_r(m, n) \leq \sum_{i=1}^b \Psi^s_r(m_i, n_i) + 2\Psi^s_r(b, n_0)G_{s-1}(m) + mH_{s-1}(m)
\]
where \( H_{s-1}(m) = 3(r - 1) + 2F_{s-1}(m) + 2(r - 1)G_{s-1}(m) \).

**Proof:** We start with a preliminary consideration. Suppose an \( r \)-regular sequence \( u \) is split into \( o \) 1-chains \( u = u_1u_2 \ldots u_o \). Then a subsequence \( v \) of \( u \) need not be \( r \)-regular but it suffices to delete at most \((r - 1)(o - 1)\) letters from \( v \) and what remains is \( r \)-regular. This consideration will be used in this proof and then again in the fourth step.

Let \( v \) be \( r \)-regular, \( \|v\| \leq n \), \( Y(r,s) \not\subseteq v \), \( v \) consists of \( m \) 1-chains and \( |v| = \Psi^s_r(m,n) \). We group 1-chains of \( v \) in \( b \) layers (the partition \( m = m_1 + \ldots + m_b \) is given) \( L_1, L_2, \ldots, L_b \) where \( L_i \) consists of \( m_i \) 1-chains. Thus \( v = L_1L_2 \ldots L_b \). We split any \( L_i \) in three subsequences \( v^1_i, v^2_i \) and \( v^3_i \), \( v^1_i \) consists of those letters that occur only in \( L_i \) (i.e. \( S(v^1_i) \cap S(L_j) = \emptyset \) for \( i \neq j \)), \( v^2_i \) consists of those that occur also before \( L_i \) and \( v^3_i \) consists of the remaining ones (i.e. do not occur before \( L_i \) but occur after \( L_i \)). Obviously

\[
\Psi^s_r(m,n) = |v| = \sum_{i=1}^b |v^1_i| + \sum_{i=1}^b |v^2_i| + \sum_{i=1}^b |v^3_i|.
\]

The upper bound on the first term in (4) is clearly

\[
\sum_{i=1}^b (\Psi^s_r(m_i,n_i) + (m_i - 1)(r - 1)) = \sum_{i=1}^b \Psi^s_r(m_i,n_i) + (m - b)(r - 1)
\]

where \( n_i = \|v^1_i\| \). We come naturally to the partition \( n = n_0 + n_1 + \ldots + n_b \), \( n_0 \) is the number of all symbols figuring in all \( v^2_i, v^3_i \). Observe that \( Y(r,s-1) \not\subseteq v^2_i, v^3_i \) for all \( i \). This fact enables us to estimate the remaining two terms in (4). We do it only for the second one, the third one is treated similarly. According to the hypothesis

\[
\sum_{i=1}^b |v^2_i| \leq \sum_{i=1}^b (F_{s-1}(m_i)m_i + G_{s-1}(m_i)||v^2_i|| + (m_i - 1)(r - 1)) \leq F_{s-1}(m)m + G_{s-1}(m)\sum_{i=1}^b ||v^2_i|| + (m - b)(r - 1).
\]

We transform any \( v^2_i \) to \( w_i \) by taking any \( a \in S(v^2_i) \) just once (the 1-chain \( w_i \) is a subsequence of \( v^2_i) \). The sequence \( w = w_1w_2 \ldots w_b \) meets (after deleting at most \((b - 1)(r - 1)\) letters) all conditions to be estimated by \( \Psi^s_r(b,n_0) \). Thus

\[
\sum_{i=1}^b ||v^2_i|| = |w| \leq \Psi^s_r(b,n_0) + (b - 1)(r - 1).
\]

We substitute all derived bounds in (4):

\[
\Psi^s_r(m,n) \leq \sum_{i=1}^b \Psi^s_r(m_i,n_i) + (m - b)(r - 1) + 2[F_{s-1}(m)m + G_{s-1}(m)(\Psi^s_r(b,n_0) + (b - 1)(r - 1)) + (m - b)(r - 1)].
\]

We got (3). \( \square \)
Step 3.

**Lemma 2.3.** Let $F_{s-1}, G_{s-1}$ and $H_{s-1}$ be as in Lemma 2.2. Then for any $m, n \geq 1, k \geq 2$

\[(5) \quad \Psi^s_r(m, n) \leq \alpha_k(m)m. H_{s-1}(m). (5G_{s-1}(m))^{k-2} + 2n.(2G_{s-1}(m))^{k-1}. \]

**Proof:** For $m \leq 4$ (5) holds because of the trivial inequality $\Psi^s_r(m, n) \leq mn$. We prove (5) induction on $k$, for $k$ fixed induction on $m$. We start with $k = 2$. It suffices to verify induction on $m$ the estimate

\[\Psi^s_r(m, n) \leq H_{s-1}(m)[\log_2 m]m + 4G_{s-1}(m)n\]

((5) for $k = 2$) using the inequality

\[\Psi^s_r(m, n) \leq \Psi^s_r([m/2], n_1) + \Psi^s_r([m/2], n_2) + 4G_{s-1}(m)n_0 + mH_{s-1}(m)\]

((3) for $b = 2$). It is left to the reader.

In case $k > 2$, $m \geq 3$ we put in (3) $b = \lceil m/\alpha_{k-1}(m) \rceil$, $m_i \leq \lceil m/b \rceil \leq \alpha_{k-1}(m)$. Thus $\alpha_k(m_i) \leq \alpha_k(m) - 1$ (according to (1)) and $b\alpha_{k-1}(b) \leq b\alpha_{k-1}(m) \leq 2m$. We estimate the term $\Psi^s_r(m_i, n_i)$ in (3) by (5) for $k, m_i$, and the term $\Psi^s_r(b, n_0)$ by (5) for $k - 1, b$. Then

\[
\Psi^s_r(m, n) \leq \sum_{i=1}^b (H_{s-1}(m_i)(5G_{s-1}(m_i))^{k-2} \alpha_k(m_i)m_i + 2(2G_{s-1}(m_i))^{k-1}n_i) + \\
(\alpha_k(m) - 1)m + 2G_{s-1}(m))^{k-1}(n - n_0) + \\
H_{s-1}(m)(5G_{s-1}(m))^{k-2} - 1)m + 2G_{s-1}(m))^{k-1}n_0 + mH_{s-1}(m) \leq \\
H_{s-1}(m)(5G_{s-1}(m))^{k-2} \alpha_k(m)m + 2G_{s-1}(m))^{k-1}n.
\]

\[\square\]

**Lemma 2.4.** For any $s \geq 4$ the inequality

\[(6) \quad \Psi^s_r(m, n) \leq m(10r)\alpha^{s-3}(m) + 4\alpha^{s-4}(m) + n(4r)\alpha^{s-3}(m) + 2\alpha^{s-4}(m) \quad m, n \geq 1\]

holds.

**Proof:** We consider the functions $\overline{F}_s, \overline{G}_s, s \geq 3$ that are defined by the following recurrent relations (we write $\overline{F}_s$ instead of $F_s(m)$, $\overline{G}_s$ instead of $G_s(m)$ and $\alpha$ instead of $\alpha(m)$ for the sake of brevity):

\[
\overline{F}_3 = 0, \overline{G}_3 = 2r \\
\overline{F}_s = 4(3(r-1) + 2\overline{F}_{s-1} + 2(r-1)\overline{G}_{s-1})(5\overline{G}_{s-1})^{\alpha-1}, \overline{G}_s = 2(2\overline{G}_{s-1})^\alpha.
\]
Induction on \( s \) shows that

\[
\Psi_s^s(m, n) \leq F_s(m)m + G_s(m)n
\]

for any \( m, n \geq 1, s \geq 3 \). Indeed, for \( s = 3 \) it follows from Step 1 and for general \( s \) we obtain this inequality from (5) where we put \( k = \alpha(m) + 1 \) and use (2). We count explicit upper bounds on both functions. Clearly \( G_s = 2.4^s + \alpha^{s-5} \ldots + \alpha \cdot (4r)^{s-3} \) for \( s \geq 5 \) and \( G_4 = 2(4r)^s \). Hence \( G_s \leq (4r)^{s-3} + 2\alpha^{s-4} \) for \( s \geq 4 \).

Further \( F_4 = 2 \left( 4r - 1 \right) (10r) \alpha \geq G_4 \) and therefore \( F_s \geq G_s \) for all \( s \geq 4 \). Thus \( F_s \leq 4(3r - 1) + 2rF_{s-1} (5F_{s-1})^{a-1} \leq 4r(5F_{s-1})^{a} \). If we solve this recurrent relation as an equation then an upper bound on \( F_s \) is obtained. We start with \( F_4 \leq 2r(10r)^{a} \) and derive

\[
F_s \leq (2r)^{a^{s-4}}(4r)^{s-5} + 1.5^{s-4} \ldots + \alpha \cdot (10r)^{s-3} \leq (10r)^{s-3} + 4\alpha^{s-4}.
\]

\( \square \)

**Step 4.**

**Lemma 2.5.**

(7) \[
f(u, n) \leq 2\|u\|, 2\|u\|^{-4}.n.(10\|u\|)^{2\|u\|^{-4}(n) + 8\|u\|^{-5}(n)}
\]

for any sequence \( u, |u| \geq 5 \).

**PROOF:** We will find the upper bound \( nE_s(n) \) \( (E_s(n) \) is a nondecreasing function) on the quantity

\[
\max\{|v| \mid v \text{ is } r\text{-regular, } \|v\| \leq n, Y(r, s) \leq v\}.
\]

It suffices because \( u \leq v \) for any \( v \in Y(\|u\|, |u| - 1) \) except \( u = aa \ldots a \) (\( i \) times) but \( f(aa \ldots a, n) = n(i - 1) \). We derive for \( E_s \) a recurrent relation. Let \( v \) be \( r\)-regular, \( \|v\| \leq n \) and \( Y(r, s) \leq v \). We split \( v = v_1v_2v_3 \ldots v_nv_n \) where \( l_1, \ldots, l_n \) are the last letters of all \( x \in S(v) \). Observe that \( Y(r, s - 1) \leq v_i \) and hence \( |v| = \sum_{i=1}^{n} |v_i| + n \leq \left( \sum_{i=1}^{n} \|v_i\| \right) E_{s-1}(n) + n \). The sum \( \sum_{i=1}^{n} \|v_i\| \) may be estimated by \( \Psi_s^s(m, n) + (n - 1)(r - 1) \) (we use the same trick as in Lemma 2.2 — replace \( v_i \) by 1-chain of the length \( \|v_i\| \)). Thus

\[
|v| \leq 2nE_{s-1}(n).(10r)^{s-3}(n) + 4\alpha^{s-4}(n)
\]

by (6). Hence we may choose

\[
E_3(n) = 2r \text{ (see Step 1)}
\]
\[
E_s(n) = 2E_{s-1}(n).(10r)^{s-3}(n) + 4\alpha^{s-4}(n).
\]

The solution of this relation is:

\[
E_s(n) = 2r.2^{s-3}.(10r)^{s-3}(n) + \alpha^{s-4}(n) + \alpha(n) + 4\alpha^{s-4}(n) + \ldots + 4.
\]

If replaced \( r \) by \( \|u\| \) and \( s \) by \( |u| - 1 \) then (7) is obtained. \( \square \)
Concluding remarks

We achieved the exponent $\alpha |u|^{-4}(n)$ in (7) by induction starting with $s = 3$. It is possible that this bound might be improved to (roughly) $\alpha \frac{1}{2} |u|(n)$ but it would require computations far more complex as in [ASS].

More interesting than the best value in (7) is perhaps the fact that $f(u, n)$ is almost linear for any sequence $u$. Hence a double induction must be used in some form whenever we want to obtain a superlinear lower bound on $f(u, n)$ (cf. [HS], [ASS], [K], [FH] and [WS]). Methods giving such “huge” functions as $n^{\frac{2}{x}}$ or $n \log \log n$ or $n \log^{*} n$ cannot be successful. It is a remarkable difference in comparison with extremal problems concerning graphs or hypergraphs (Turán theory). Here most common functions are $n^\beta$, $\beta > 1$. A certain hybrid occurs in Davenport-Schinzel theory of matrices in [FH] where the maximum number of 1’s in a 0-1 matrix (of the size $n \times n$) which does not contain a forbidden subconfiguration is investigated. Here $n^{\alpha(n)}$ figurates as an upper bound as well as $n^{\frac{4}{x}}$ and $n \log n$.

For obtaining a good general upper bound on $f(u, n)$ only basic features of $u$ — such as the length and the number of symbols — were important. It is demonstrated by the fact that we worked instead of $u$ itself with the sets $X(k, l)$ resp. $Y(k, l)$ that are determined by $|u|$ and $\|u\|$. It is probable that this changes if we start to investigate finer properties of the asymptotic growth of $f(u, n)$. But except for the case $u = al_{s}$ where we know the magnitude of $f(u, n)$ with high precision due the deep result of [ASS] only little about that function is known. One of the basic questions is to determine the set

$$Lin = \{ u \mid f(u, n) = O(n) \}$$

— see [AKV] and [KL] for a partial solution.

References


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