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Bifurcation for some semilinear elliptic equations when the linearization has no eigenvalues

WOLFGANG ROTHER

Abstract. We prove existence and bifurcation results for a semilinear eigenvalue problem in \mathbb{R}^N ($N \geq 2$), where the linearization $-\Delta$ has no eigenvalues. In particular, we show that under rather weak assumptions on the coefficients $\lambda = 0$ is a bifurcation point for this problem in H^1, H^2 and L^p ($2 \leq p \leq \infty$).

Keywords: bifurcation point, variational method, eigenvalues, exponential decay, standing waves

Classification: 35P30, 35A30

1. Introduction and presentation of the results.

In the present paper, we consider the nonlinear eigenvalue problem

$$(1.1) \quad -\Delta u - q(x)|u|^{\sigma_1}u + r(x)|u|^{\sigma_2}u = \lambda u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 2$ and σ_1 and σ_2 are positive constants such that $\sigma_1 < 4/N$. In particular, we are interested in the question if $\lambda = 0$ is a bifurcation point for the equation (1.1).

Since the problem (1.1) is considered in \mathbb{R}^N , the linearization $-\Delta$ has no eigenvalues and $\lambda = 0$ is the infimum of the spectrum of $-\Delta$. In case that $r \equiv 0$, this problem has been studied by many authors. See for instance [5]–[7], [9], [13]–[18] and the literature quoted therein. In case that $r \not\equiv 0$, we only know some existence results for the equation (1.1) (see [1], [2], [8] and [12]), but no bifurcation results. In the following, we will close this gap by presenting some bifurcation results for the general case.

We always assume that the functions q and r satisfy the subsequent conditions:

(A) The functions $q, r : \mathbb{R}^N \rightarrow \mathbb{R}$ are measurable and r fulfills $r(x) \geq 0$ for almost all $x \in \mathbb{R}^N$.

(B) There exist a constant $0 < a \leq 2 - (\sigma_1 N/2)$ and an open ball $B \subset \mathbb{R}^N$, satisfying $B \neq \emptyset$ and $0 \notin \bar{B}$ (\bar{B} is the closure of B), such that $q(x) \geq f(x)|x|^{-a}$ holds for almost all $x \in \zeta$, where $\zeta = \{tx; t \geq 1, x \in B\}$ and $f : \zeta \rightarrow [0, \infty)$ is a measurable function satisfying $f(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.

Moreover, we assume that there exists a constant \mathcal{K} such that

$$r(x) \leq \mathcal{K}|x|^b \quad \text{holds for almost all } x \in \zeta,$$

where b is defined by $b = (2 - a)(\sigma_2/\sigma_1) - 2$.

(C) The functions r and $q_- = \min(q, 0)$ are locally integrable.

(D) The function $q_+ = \max(q, 0)$ can be written as $q_+ = q_1 + q_2$, where

(D1) the function q_1 satisfies $0 \leq q_1 \in L^\infty$, and $q_1(x)$ tends uniformly to zero as $|x| \rightarrow \infty$,

(D2) and the function q_2 satisfies $0 \leq q_2 \in L^{p_0}$ for some constant

$$2N/(4 - \sigma_1 N) < p_0 < \infty.$$

We want to point out that the above assumptions allow the function q to decay exponentially to $-\infty$ or faster in some direction, and allow the function r to increase exponentially to $+\infty$ or faster in some direction.

Theorem 1.1. *Suppose that the functions q and r satisfy the assumptions (A)–(D) and that the constant a is defined as in condition (B). Then, there exists a constant $\mu_a \in (0, \infty]$, depending on a , such that for each $\mu \in (0, \mu_a)$ there exists a nonpositive constant $\lambda(\mu)$ and a nontrivial nonnegative function $u_\mu \in H^1 \cap L^\infty$ which solves equation (1.1) in the sense of distributions. In case that $a = 2 - (\sigma_1 N/2)$, we have $\mu_a = \infty$. Moreover, it follows that $\lambda(\mu) \rightarrow 0$, $\|u_\mu\|_{H^1} \rightarrow 0$ and, if $p \in [2, \infty]$, that $\|u_\mu\|_p \rightarrow 0$ as $\mu \rightarrow 0$. Hence, $\lambda = 0$ is a bifurcation point for equation (1.1) in H^1 and in L^p for $p \in [2, \infty]$.*

Corollary 1.2. (a) *If $q_-, r \in L^p_{\text{loc}}$ holds for some constant $p > N/2$, then u_μ is positive and locally Hölder continuous.*

(b) *If q and r are locally Hölder continuous, then we have $u_\mu \in C^2$ and the equation (1.1) holds in the classical sense.*

Corollary 1.3. *Suppose in addition to (A)–(D) that $p_0 \geq 2$ and that $q, r \in L^\infty + L^2$. Then, it follows that $u_\mu \in H^2$ and that $\|u_\mu\|_{H^2} \rightarrow 0$ as $\mu \rightarrow 0$. Thus, $\lambda = 0$ is a bifurcation point for (1.1) in H^2 .*

Remark 1.4. In case that $r \equiv 0$, Corollary 1.3 improves Theorem 2.6 (c) in [13]. In [13] it is assumed that q is nonnegative, that $q = q_+$ satisfies condition (D) and that $p_0 \geq 2$. Moreover, it is assumed

(i) that there exist constants $A > 0$ and $0 \leq t < 2 - (\sigma_1 N/2)$ such that $q(x) \geq A(1 + |x|)^{-t}$ holds a.e. in \mathbb{R}^N . In case that $N \geq 3$ the author requires additionally

(ii) that $\sigma_1 < 2/(N - 2)$ and $p_0 > 2N/(2 - \sigma_1(N - 2))$. Hence, Corollary 1.3 shows that the condition (i) can be weakened considerably and that condition (ii) is superfluous.

The solutions of the equation (1.1) supply standing waves for nonlinear Klein-Gordon and Schrödinger equations. So, from the standpoint of physics it is an interesting question if the solutions of (1.1) decay exponentially to 0 at infinity.

For the proof of the exponential decay to 0 we need an additional assumption:

(E) There exists a constant $R_0 > 0$ such that q_2 satisfies

$$q_2(x) = 0 \text{ for almost all } |x| \geq R_0.$$

Theorem 1.5. *Suppose that $\sigma_2 \leq \sigma_1$ and that the functions q and r satisfy the assumptions (A)–(E). Then, for each $\mu \in (0, \mu_a)$ the function u_μ decays exponentially to 0 at infinity.*

Theorem 1.6. *Suppose that $\sigma_1 < \sigma_2$ and that the functions q and r satisfy the assumptions (A)–(E). Then, there exists a decreasing sequence $(\mu_n) \subset (0, \mu_a)$ such that $\lim_{n \rightarrow \infty} \mu_n = 0$ and u_{μ_n} decays exponentially to 0 at infinity.*

The proofs for Theorem 1.5–1.6 can be found in § 4.

2. Some preliminaries.

For $p \in [1, \infty]$, $L^p = L^p(\mathbb{R}^N)$ and $L^p_{\text{loc}} = L^p_{\text{loc}}(\mathbb{R}^N)$ are the usual Lebesgue spaces and $\|\cdot\|_p$ is the norm on L^p . If $1 < p < \infty$, then the dual index p' of p is defined by $p' = p/(p - 1)$. Furthermore, H^k ($k = 1, 2$) is the Hilbert space $H^k(\mathbb{R}^N) = W^{k,2}(\mathbb{R}^N)$. The norm on H^1 is given by $\|u\|_{H^1} = (\|\nabla u\|_2^2 + \|u\|_2^2)^{1/2}$ and the norm on H^2 by $\|u\|_{H^2} = (\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2)^{1/2}$. Finally, $C_0^\infty = C_0^\infty(\mathbb{R}^N)$ denotes the set of all functions which have compact support and derivatives of any order.

If $N = 2$, then it follows from the Sobolev imbedding theorem that for each $p \in [2, \infty)$ there exists a constant A_p such that

$$(2.1) \quad \|u\|_p \leq A_p \|u\|_{H^1} \quad \text{holds for all } u \in H^1.$$

In case that $N \geq 3$, we define $2^* = 2N/(N - 2)$. Then, there exists a constant C_0 such that

$$(2.2) \quad \|u\|_{2^*} \leq C_0 \|\nabla u\|_2 \quad \text{holds for all } u \in H^1.$$

In particular we see that for each $p \in [2, 2^*]$ there exists a constant B_p such that

$$(2.3) \quad \|u\|_p \leq B_p \|u\|_{H^1} \quad \text{holds for all } u \in H^1.$$

Let F be one of the Banach spaces H^1 , H^2 or L^p . Then a real number λ is called a bifurcation point for the equation (1.1) in F if and only if there exists a sequence $(\lambda_n, u_n) \subset \mathbb{R} \times F$ such that $u_n \not\equiv 0$, $\lambda_n \rightarrow \lambda$, $\|u_n\|_F \rightarrow 0$ ($n \rightarrow \infty$) and

$$\int \nabla u_n \nabla \varphi \, dx - \int q|u_n|^{\sigma_1} u_n \varphi \, dx + \int r|u_n|^{\sigma_2} u_n \varphi \, dx = \lambda_n \int u_n \varphi \, dx$$

holds for all $\varphi \in C_0^\infty$ and $n \in \mathbb{N}$.

When the domain of integration is not indicated, it is understood to be \mathbb{R}^N .

Lemma 2.1. *Let $v \in H^1$ be a nonnegative function. Then, there exists a sequence (φ_n) of nonnegative functions $\varphi_n \in C_0^\infty$ such that*

$$\varphi_n \rightarrow v \quad \text{in } H^1.$$

PROOF: The functions η_n ($n \in \mathbb{N}$) may be chosen such that $\eta_n \in C_0^\infty$, $0 \leq \eta_n \leq 1$, $\eta_n(x) = 1$ holds for $|x| \leq n$, $\eta_n(x) = 0$ if $|x| \geq n + 1$ and $\|\nabla \eta_n\|_\infty \leq C$, where the constant C is independent of n . Then $\eta_n v \rightarrow v$ in H^1 .

For a function $u \in L^1_{\text{loc}}$, the regularization u_ε may be defined as in [3, p. 147]. Then, we can find a sequence (ε_n) of positive numbers ε_n , satisfying $\varepsilon_n \rightarrow 0$, such that $\varphi_n = (\eta_n v)_{\varepsilon_n} \rightarrow v$ in H^1 . □

Lemma 2.2. *Let $v \in H^1$ be a nonnegative function and, for $t > 0$, v_t may be defined by $v_t = \min(v, t)$. Then it follows that $v_t \in H^1$, $\partial_i v_t = \partial_i v$ holds almost everywhere in $\{x; v(x) \leq t\}$ and $\partial_i v_t = 0$ holds almost everywhere in $\{x; v(x) > t\}$. Moreover, for each $s \in [1, \infty)$, we have $0 \leq v_t^s \in H^1 \cap L^\infty$ and $\partial_i v_t^s = s v_t^{s-1} \partial_i v_t$ ($i = 1, \dots, N$).*

PROOF: The first part of the lemma follows from Lemma 1.1 in [10] and Theorem 7.8 in [3]. The functions η_n and the regularizations u_ε may be defined as in the proof of Lemma 2.1. Then, there exists a sequence of positive numbers (ε_n) such that $\varepsilon_n \rightarrow 0$ and

$$\varphi_n = (\eta_n v_t)_{\varepsilon_n} \longrightarrow v_t \text{ in } H^1.$$

Here, the functions φ_n satisfy $\varphi_n \in C_0^\infty$ and $0 \leq \varphi_n \leq t$. Since $\varphi_n \rightarrow v_t$ in L^2 , we can find a subsequence $(\varphi_{n(k)})$ of (φ_n) such that $\varphi_{n(k)}(x) \rightarrow v_t(x)$ for almost all $x \in \mathbb{R}^N$.

Now, suppose that $s > 1$. Then it follows that $\varphi_{n(k)}^s \in C_0^1$ and that

$$\partial_i \varphi_{n(k)}^s = s \varphi_{n(k)}^{s-1} \partial_i \varphi_{n(k)}.$$

Moreover, since $|v_t^s - \varphi_{n(k)}^s| \leq s |v_t - \varphi_{n(k)}| t^{s-1}$, we see that $\varphi_{n(k)}^s \rightarrow v_t^s$ in L^2 . Hence, we obtain: $\partial_i v_t^s = s v_t^{s-1} \partial_i v_t$. □

The following lemma can be found in [11, p. 93].

Lemma 2.3. *Suppose that $\varphi(t)$ ($t \in [t_0, \infty)$) is a nonnegative and nonincreasing function such that $\varphi(h) \leq C(h - t)^{-\gamma} \varphi(t)^\delta$ holds for all $h > t \geq t_0$. The constants γ and C are assumed to be positive and δ may satisfy $\delta > 1$. Then, for $d = C^{1/\gamma} \varphi(t_0)^{(\delta-1)/\gamma} 2^{\delta/(\delta-1)}$ it follows that $\varphi(t_0 + d) = 0$.*

3. Proof of the main results.

In the present paragraph, we will prove Theorem 1.1 and Corollary 1.2–1.3. We start with

Lemma 3.1. *There exist positive constants α and β , and for each $\varepsilon > 0$ a constant $K_\varepsilon > 0$, such that*

$$(2 + \sigma_1)^{-1} \int q_+ |u|^{2+\sigma_1} dx \leq \varepsilon \|\nabla u\|_2^2 + K_\varepsilon \left(\|u\|_2^{2+\alpha} + \|u\|_2^{2+\beta} \right)$$

holds for all $u \in H^1$.

PROOF: For $\varepsilon = \frac{1}{4}$, the proof can be found in [5, pp. 568–569]. For general $\varepsilon > 0$, the proof proceeds quite similarly. □

The nonlinear functional ξ may be defined by

$$\begin{aligned} \xi(u) = \frac{1}{2} \int |\nabla u|^2 dx - (2 + \sigma_1)^{-1} \int q |u|^{2+\sigma_2} dx \\ + (2 + \sigma_2)^{-1} \int r |u|^{2+\sigma_2} dx. \end{aligned}$$

By D , we denote the set

$$D = \{u \in H^1; \int |q_-||u|^{2+\sigma_1} dx < \infty \text{ and } \int r|u|^{2+\sigma_2} dx < \infty\}.$$

Moreover, for $\mu \geq 0$, we define $D_\mu = \{u \in D; \|u\|_2 \leq \mu\}$. Then, according to Lemma 3.1, we see that $I(\mu) = \inf_{u \in D_\mu} \xi(u)$ is a well defined real number.

Lemma 3.2. (a) *Suppose that the constant a in condition (B) satisfies $a = 2 - (\sigma_1 N/2)$. Then it follows that $I(\mu) < 0$ holds for all $\mu > 0$.*

(b) *Suppose that $a < 2 - (\sigma_1 N/2)$. Then, there exists a constant $\mu_a > 0$ such that $I(\mu) < 0$ holds for all $\mu \in (0, \mu_a)$.*

Remark 3.3. In the following, we define $\mu_a = \infty$ if $a = 2 - (\sigma_1 N/2)$.

PROOF OF LEMMA 3.2: The ball B may be defined as in condition (B) and ν may be a positive constant. Then, the function $\varphi_0 \in C_0^\infty$ may be chosen such that $\text{supp } \varphi_0 \subset B$ and $\|\varphi_0\|_2 = \nu$. Moreover, for each $t \geq 1$, we define $\varphi_t(x) = t^k \varphi_0(t^{-1}x)$, where $k = (a - 2)/\sigma_1$. Since $\|\varphi_t\|_2 = \nu t^{k+(N/2)}$, we see that $\varphi_t \in D_{\nu t^{k+(N/2)}}$ and that

$$\begin{aligned} I\left(\nu t^{k+(N/2)}\right) &\leq \xi(\varphi_t) = t^{2k+N-2} \left(\frac{1}{2} \int |\nabla \varphi_0(x)|^2 dx \right. \\ &\quad - t^{2+k\sigma_1} (2 + \sigma_1)^{-1} \int_B q(tx) |\varphi_0(x)|^{2+\sigma_1} dx \\ &\quad \left. + t^{2+k\sigma_2} (2 + \sigma_2)^{-1} \int_B r(tx) |\varphi_0(x)|^{2+\sigma_2} dx \right) \\ &\leq t^{2k+N-2} \left(\frac{1}{2} \int |\nabla \varphi_0(x)|^2 dx \right. \\ &\quad - \inf_{x \in B} f(tx) (2 + \sigma_1)^{-1} \int_B |x|^{-a} |\varphi_0(x)|^{2+\sigma_1} dx \\ &\quad \left. + \mathcal{K} (2 + \sigma_2)^{-1} \int_B |x|^b |\varphi_0(x)|^{2+\sigma_2} dx \right). \end{aligned}$$

Since $\inf_{x \in B} f(tx) \rightarrow \infty$ as $t \rightarrow \infty$, we can find a constant $t_0 \geq 1$ such that

$$(3.1) \quad I\left(\nu t^{k+(N/2)}\right) < 0 \text{ holds for all } t > t_0.$$

Now, suppose that $a = 2 - (\sigma_1 N/2)$. Then, we have $k + (N/2) = 0$. Hence, the part (a) of the lemma follows from (3.1) for $\nu = \mu$. In case that $a < 2 - (\sigma_1 N/2)$, we have $k + (N/2) < 0$. Then, the assertion of the part (b) follows from (3.1) if we define $\nu = 1$, $\mu_a = t_0^{k+(N/2)}$ and $\mu = t^{k+(N/2)}$. □

Lemma 3.4. *For each $\mu \in (0, \mu_a)$ there exists a function $u_\mu \in D_\mu$ such that $u_\mu \geq 0$, $\|u_\mu\|_2 > 0$ and $\xi(u_\mu) = I(\mu)$.*

PROOF: Let $\mu \in (0, \mu_a)$, and $(v_n) \subset D$ may be a sequence such that $\xi(v_n) \rightarrow I(\mu)$. Then, we may assume without restriction that $\xi(v_n) \leq 0$ and that $v_n \geq 0$ holds for all n . Hence, we obtain from Lemma 3.1:

$$(3.2) \quad \begin{aligned} & \frac{1}{4} \|\nabla v_n\|_2^2 + (2 + \sigma_1)^{-1} \int |q_-| |v_n|^{2+\sigma_1} dx \\ & + (2 + \sigma_2)^{-1} \int r |v_n|^{2+\sigma_1} dx \leq K_{1/4} (\mu^{2+\alpha} + \mu^{2+\beta}). \end{aligned}$$

Since (v_n) is bounded in H^1 , we can find a subsequence of (v_n) , still denoted by (v_n) , and a $u_\mu \in H^1$ such that $v_n \xrightarrow{w} u_\mu$ in H^1 and $v_n(x) \rightarrow u_\mu(x)$ for almost all $x \in \mathbb{R}^N$. Then, it follows from the uniform boundedness principle, (3.2) and Fatou's lemma that $\|u_\mu\|_2 \leq \mu$, $\|\nabla u_\mu\|_2 \leq \liminf \|\nabla v_n\|_2$,

$$\int |q_-| |u_\mu|^{2+\sigma_1} dx \leq \liminf \int |q_-| |v_n|^{2+\sigma_1} dx < \infty$$

and

$$\int r |u_\mu|^{2+\sigma_2} dx \leq \liminf \int r |v_n|^{2+\sigma_2} dx < \infty.$$

Moreover, we see that $u_\mu \geq 0$. Since the imbedding $H^1(G) \rightarrow L^{(2+\sigma_1)p'_0}(G)$ is compact for all bounded balls G and $q_1(x) \rightarrow 0$ as $|x| \rightarrow \infty$, it follows that

$$\int q_+ |v_n|^{2+\sigma_1} dx \longrightarrow \int q_+ |u_\mu|^{2+\sigma_1} dx \quad (\text{see [5, p. 570]}).$$

Moreover, we obtain

$$I(\mu) \leq \xi(u_\mu) \leq \liminf \xi(v_n) = I(\mu) < 0$$

and consequently that $\xi(u_\mu) = I(\mu)$ and $\|u_\mu\|_2 > 0$. □

Lemma 3.5. *For $\mu \in (0, \mu_a)$, the function u_μ may be chosen as in Lemma 3.4. Then, it follows that*

$$\int \nabla u_\mu \nabla \varphi dx - \int q |u_\mu|^{\sigma_1} u_\mu \varphi dx + \int r |u_\mu|^{\sigma_2} u_\mu \varphi dx = \lambda(\mu) \int u_\mu \varphi dx$$

holds for all functions $\varphi \in C_0^\infty$, where

$$\lambda(\mu) = \|u_\mu\|_2^{-2} \left(\|\nabla u_\mu\|_2^2 - \int q |u_\mu|^{2+\sigma_1} dx + \int r |u_\mu|^{2+\sigma_2} dx \right).$$

PROOF: Let $\varphi \in C_0^\infty$. Then $d\xi(\|u_\mu\|_2 \|u_\mu + \varepsilon \varphi\|_2^{-1} (u_\mu + \varepsilon \varphi)) / d\varepsilon |_{\varepsilon=0} = 0$ implies the assertion. □

Lemma 3.6. *The constant $\lambda(\mu)$ may be defined as in Lemma 3.5. Then, we have $\lambda(\mu) \leq 0$.*

PROOF: For all $t \in (0, 1]$, we have

$$\xi(u_\mu) = I(\mu) \leq I(t\mu) \leq \xi(tu_\mu).$$

Hence $\lambda(\mu) = \|u_\mu\|_2^{-2} d\xi(tu_\mu)/dt |_{t=1} \leq 0$ implies the assertion. □

Proposition 3.7. *The constants α and β may be chosen as in Lemma 3.1. Then, there exists a constant C such that*

$$|\lambda(\mu)| \leq C(\mu^\alpha + \mu^\beta) \quad \text{and} \quad \|\nabla u_\mu\|_2^2 \leq C(\mu^{2+\alpha} + \mu^{2+\beta})$$

holds for all $\mu \in (0, \mu_a)$. Hence, $\lambda = 0$ is a bifurcation point for the equation (1.1) in H^1 .

PROOF: Since $\xi(u_\mu) < 0$, we obtain from Lemma 3.1 that

$$(3.3) \quad \|\nabla u_\mu\|_2^2 \leq 4K_{1/4}(\|u_\mu\|_2^{2+\alpha} + \|u_\mu\|_2^{2+\beta}) \leq 4K_{1/4}(\mu^{2+\alpha} + \mu^{2+\beta}).$$

Moreover, since $\lambda(\mu) \leq 0$, it follows from (3.3) and Lemma 3.1 that

$$\begin{aligned} |\lambda(\mu)| &= -\lambda(\mu) \leq \|u_\mu\|_2^{-2} \int q_+ |u_\mu|^{2+\sigma_1} dx \\ &\leq (2 + \sigma_1)(4K_{1/4} + K_1) \left(\|u_\mu\|_2^\alpha + \|u_\mu\|_2^\beta \right) \leq C(\mu^\alpha + \mu^\beta). \end{aligned}$$

□

Lemma 3.8. *For all nonnegative functions $v \in H^1$ we obtain*

$$(3.4) \quad \int \nabla u_\mu \nabla v \, dx \leq \lambda(\mu) \int u_\mu v \, dx + \int q_+ u_\mu^{1+\sigma_1} v \, dx$$

and, according to Lemma 3.6, that

$$(3.5) \quad \int \nabla u_\mu \nabla v \, dx \leq \int q_+ u_\mu^{1+\sigma_1} v \, dx.$$

PROOF: Clearly, the assertion holds for all nonnegative functions $v \in C_0^\infty$. Hence, the result follows from Lemma 2.1. □

Lemma 3.9. *Suppose that $N \geq 3$ and that $\int q_+ u_\mu^{1+\sigma_1+s} \, dx < \infty$ holds for some constant $s > 1$. Then, it follows that $u_\mu \in L^{2^*(s+1)/2}$.*

PROOF: For $t > 0$, the function v_t may be defined by $v_t = \min(u_\mu, t)$. Then, according to Lemma 2.2, we see that $0 \leq v_t^s \in H^1$. Inserting v_t^s in (3.5) shows that

$$4s(s+1)^{-2} \int |\nabla v_t^{(s+1)/2}|^2 \, dx \leq \int q_+ u_\mu^{1+\sigma_1+s} \, dx.$$

Hence, using (2.2) and letting $t \rightarrow \infty$, we obtain the assertion by Fatou's lemma. □

Lemma 3.10. For each $p \in [2, \infty)$, we have $u_\mu \in L^p$.

PROOF: For $N = 2$ and for $p \in [2, 2^*]$, if $N \geq 3$, the assertion follows from the Sobolev imbedding theorem. Now, suppose that $N \geq 3$ and that the constants r_n and s_n are defined by $r_n = 2^*(1 + \varepsilon_0)^n$ and $s_n = (r_n/p'_0) - 1 - \sigma_1$, where $\varepsilon_0 = (2^*/2p'_0) - (\sigma_1/2) - 1$. Here, the constant p_0 is defined as in condition (D2). Since $p_0 > 2N/(4 - \sigma_1N + 2\sigma_1)$ and $r_n \geq 2^*$, it follows that $\varepsilon_0 > 0$ and $s_n > 1$.

Now, assume that $u_\mu \in L^{r_n}$ holds for some $n \in \mathbb{N}_0$. Then $2 \leq 1 + \sigma_1 + s_n < (1 + \sigma_1 + s_n)p'_0 = r_n$ implies that

$$\int q_+ u_\mu^{1+\sigma_1+s_n} dx < \infty.$$

Hence, we obtain from Lemma 3.9 that $u_\mu \in L^{2^*(s_n+1)/2}$. But

$$\begin{aligned} (2^*/2)(s_n + 1) &= (2^*/2)((r_n/p'_0) - \sigma_1) \\ &\geq (2^*/2)(r_n/p'_0) - (r_n/2)\sigma_1 \\ &= r_n(1 + \varepsilon_0) = r_{n+1} \end{aligned}$$

implies that $u_\mu \in L^{r_{n+1}}$. Hence, we see that $u_\mu \in L^p$ holds for all $p \in [2^*, \infty)$. \square

Lemma 3.11. For each $\mu \in (0, \mu_a)$, we have $u_\mu \in L^\infty$.

PROOF: For $t > 0$, we define the function U_t by $U_t = (u_\mu - t)_+$ and the set $A(t)$ by $A(t) = \{x; u_\mu(x) \geq t\}$. Then, we obtain from (3.5) that

$$(3.6) \quad \int \nabla u_\mu \nabla U_t dx \leq \int_{A(t)} q_+ u_\mu^{2+\sigma_1} dx.$$

The constant p_1 may be defined by $p_1 = 2N/(4 - \sigma_1N)$. Since $p_0 > p_1$, we can find a constant $p_2 \in (1, \infty)$ such that $1/p'_0 \cdot 1/p'_2 = 1/p'_1$. Then, the inequality (3.6) implies

$$(3.7) \quad \int |\nabla U_t|^2 dx \leq C(\mu)(\text{meas } A(t))^{1/p'_1}$$

for all $t > 0$, where $C(\mu)$ is defined by

$$(3.8) \quad \begin{aligned} C(\mu) &= \|q_1\|_\infty \left(\int u_\mu^{(2+\sigma_1)p_1} dx \right)^{1/p_1} \\ &\quad + \|q_2\|_{p_0} \left(\int u_\mu^{(2+\sigma_1)p'_0 p_2} dx \right)^{1/(p'_0 p_2)}. \end{aligned}$$

Now, let us assume that $N \geq 3$. Then, it follows from (2.2) and (3.7) that

$$(3.9) \quad \left(\int_{A(t)} (u_\mu - t)^{2^*} dx \right)^{2/2^*} \leq C_0^2 C(\mu) (\text{meas } A(t))^{1/p'_1}.$$

Moreover, for each $h > t$, we have

$$(3.10) \quad \begin{aligned} \left(\int_{A(t)} (u_\mu - t)^{2^*} dx \right)^{2/2^*} &\geq \left(\int_{A(h)} (u_\mu - t)^{2^*} dx \right)^{2/2^*} \\ &\geq (h - t)^2 (\text{meas } A(h))^{2/2^*}. \end{aligned}$$

Combining (3.9) and (3.10) yields

$$\text{meas } A(h) \leq (C_0^2 C(\mu))^{2^*/2} (h - t)^{-2^*} (\text{meas } A(t))^{2^*/2p'_1}$$

for all $h > t > 0$. Since $2^*/(2p'_1) = 1 + (\sigma_1 N)/2(N - 2) > 1$, it follows from Lemma 2.3 that u_μ is essentially bounded. Moreover, for each $t_0 > 0$, we have

$$\|u_\mu\|_\infty \leq d + t_0,$$

where $d = C_0 C(\mu)^{1/2} (\text{meas } A(t_0))^{\sigma_1/4} 2^{1+(2(N-2)/\sigma_1 N)}$. For $t_0 = \|u_\mu\|_2$, it follows that

$$\text{meas } A(t_0) \leq \|u_\mu\|_2^{-2} \int_{A(t_0)} u_\mu^2 dx \leq 1.$$

Hence, we obtain that

$$(3.11) \quad \|u_\mu\|_\infty \leq C_0 C(\mu)^{1/2} 2^{1+(2(N-2)/\sigma_1 N)} + \mu.$$

Finally, we consider the case that $N = 2$. Here, we obtain for all $t > 0$:

$$(3.12) \quad \begin{aligned} \int U_t^2 dx &\leq \int_{A(t)} u_\mu^2 dx \\ &\leq \left(\int_{A(t)} u_\mu^{2p_1} dx \right)^{1/p_1} (\text{meas } A(t))^{1/p'_1}. \end{aligned}$$

Combining (3.7) and (3.12) yields

$$\|U_t\|_{H^1}^2 \leq C^*(\mu) (\text{meas } A(t))^{1/p'_1}$$

for all $t > 0$, where

$$(3.13) \quad C^*(\mu) = C(\mu) + \left(\int u_\mu^{2p_1} dx \right)^{1/p_1}.$$

Hence, (2.1) implies

$$\left(\int_{A(t)} (u_\mu - t)^p dx \right)^{2/p} \leq C_p^2 C^*(\mu) (\text{meas } A(t))^{1/p'_1}$$

for all $t > 0$ and $p \in [2, \infty)$. Then, proceeding as in the case that $N \geq 3$, one can show that

$$\text{meas } A(h) \leq C_p^p C^*(\mu)^{p/2} (h - t)^{-p} (\text{meas } A(t))^{p/(2p'_1)}$$

holds for all $h > t > 0$ and $p \in [2, \infty)$. Hence, according to Lemma 2.3, we see that u is essentially bounded and that

$$(3.14) \quad \|u_\mu\|_\infty \leq C_p C^*(\mu)^{1/2} 2^{(p/(2p'_1))((p/2p'_1)-1)} + \mu$$

if $p > 2p'_1$. □

Lemma 3.12. *For all $p \in [2, \infty)$ we have $\|u_\mu\|_p \rightarrow 0$ as $\mu \rightarrow 0$.*

PROOF: We start with the case that $N = 2$. Then, according to (2.1), we obtain:

$$\|u_\mu\|_p \leq C_p \|u_\mu\|_{H^1} \quad \text{for all } \mu \in (0, \mu_a).$$

Hence, the assertion follows from Proposition 3.7. In case that $N \geq 3$ and $p \in [2, 2^*]$, the assertion is obtained by (2.3) and Proposition 3.7. Now, assume that $N \geq 3$ and that $p \in (2^*, \infty)$. Then, we can find a constant $t > 0$ such that $p = (1 + (t/2))2^*$. Thus, by the Sobolev inequality (2.2), we see that

$$\begin{aligned} (3.15) \quad \|u_\mu\|_p^{2+t} &= \|u_\mu^{1+(t/2)}\|_{2^*}^2 \leq C_0^2 \|\nabla u_\mu^{1+(t/2)}\|_2^2 \\ &= C_0^2 (1 + (t/2))^2 (1+t)^{-1} \int \nabla u_\mu \nabla u_\mu^{1+t} dx. \end{aligned}$$

The right hand side of (3.15) is well defined since u_μ is bounded. From (3.5), we conclude that

$$\begin{aligned} (3.16) \quad \int \nabla u_\mu \nabla u_\mu^{1+t} dx &\leq \int q_+ u_\mu^{2+\sigma_1+t} dx \\ &\leq \|q_1\|_\infty \int u_\mu^{2+\sigma_1+t} dx + \|q_2\|_{p_0} \left(\int u_\mu^{(2+\sigma_1+t)p'_0} dx \right)^{1/p'_0}. \end{aligned}$$

Since

$$\begin{aligned} p'_0 &< 2N/(2(N-2) + \sigma_1 N) < 2N/(2(N-2) + \sigma_1(N-2)) \\ &\leq (2N + tN)/((2 + \sigma_1)(N-2) + t(N-2)) \\ &= (2 + \sigma_1 + t)^{-1} \cdot (2N + tN)/(N-2) \\ &= (2 + \sigma_1 + t)^{-1} p, \end{aligned}$$

we see that there is a constant $\tau \in (0, 1)$ such that

$$(2 + \sigma_1 + t)p'_0 = \tau p + (1 - \tau)2.$$

Hence, by Hölder's inequality, we obtain

$$\left(\int u_\mu^{(2+\sigma_1+t)p'_0} dx \right)^{1/p'_0} \leq \|u_\mu\|_p^{\tau p/p'_0} \|u_\mu\|_2^{2(1-\tau)/p'_0}.$$

Then, using again the fact that $p'_0 < 2N/(2(N-2) + \sigma_1 N)$, it is not difficult to show that $p\tau/p'_0 < 2 + t$.

Quite similarly, one can prove that there exist constants $c_1 \in (0, 2+t)$ and $c_2 > 0$ such that $\int u_\mu^{2+\sigma_1+t} dx \leq \|u_\mu\|_p^{c_1} \|u_\mu\|_2^{c_2}$. Hence, we conclude from (3.15), (3.16) and Young's inequality that $\|u_\mu\|_p \rightarrow 0$ as $\mu \rightarrow 0$. \square

Lemma 3.13. *We have $\|u_\mu\|_\infty \rightarrow 0$ as $\mu \rightarrow 0$.*

PROOF: The constants $C(\mu)$ and $C^*(\mu)$ may be defined as in (3.8) and (3.13). Then, according to Lemma 3.12, it follows that $C(\mu) \rightarrow 0$ and $C^*(\mu) \rightarrow 0$ as $\mu \rightarrow 0$. Hence, the assertion follows from (3.11) and (3.14). \square

PROOF OF COROLLARY 1.2: Suppose that the assumptions of part (a) are fulfilled. Then, according to Lemma 3.5, we see that

$$-\Delta u_\mu + c(x)u_\mu = 0 \text{ holds in } \mathcal{D}'(\mathbb{R}^N),$$

where $c(x) = -q(x)u_\mu^{\sigma_1}(x) + r(x)u_\mu^{\sigma_2}(x) - \lambda(\mu)$. Since $p_0 > N/2$ and $u_\mu \in L^\infty$, we see that $c \in L_{loc}^{p_1}$, where $p_1 = \min(p_0, p)$ satisfies $p_1 > N/2$. Now, the assertion follows from Theorem 7.1 and Corollary 8.1 in [10].

Next, we suppose that the assumptions of the part (b) are fulfilled. Then, it follows from part (a) that u is locally Hölder continuous. Hence, the distribution Δu_μ can be represented by a locally Hölder continuous function. Thus, the assertion of the part (b) follows by a well known result from the regularity theory of elliptic differential equations. \square

PROOF OF COROLLARY 1.3: According to Lemma 3.5, we see that

$$(3.17) \quad -\Delta u_\mu = \lambda(\mu)u_\mu + qu_\mu^{1+\sigma_1} - ru_\mu^{1+\sigma_2} \text{ holds in } \mathcal{D}'(\mathbb{R}^N).$$

Then, it follows from the assumptions and from Lemma 3.10 – Lemma 3.13 that the right hand side of (3.17) defines a function $F_\mu \in L^2$ such that $\|F_\mu\|_2 \rightarrow 0$ as $\mu \rightarrow 0$. Consequently, we see that $u_\mu \in H^2$ and that $\|u_\mu\|_{H^2} \rightarrow 0$ as $\mu \rightarrow 0$. \square

4. Exponential decay.

Lemma 4.1. *Suppose that the functions q and r satisfy the assumptions (A)–(E) and that for $\mu \in (0, \mu_a)$ the function u_μ and the constant $\lambda(\mu)$ are defined as in Lemma 3.4 resp. Lemma 3.5. Moreover, we assume that $\lambda(\mu) < 0$ holds for some $\mu \in (0, \mu_a)$. Then, for each $c \in (0, -\lambda(\mu))$ there exists a constant A_c such that*

$$u_\mu(x) \leq A_c \exp(-(-\lambda(\mu) - c)^{1/2}|x|)$$

holds for almost all $x \in \mathbb{R}^N$.

PROOF: Using the fact that u_μ is bounded, we conclude from (D1) and (E) that there exists a constant $R_c > R_0$ such that

$$(4.1) \quad q_+(x)u_\mu^{\sigma_1}(x) \leq c \text{ holds for almost all } x \in \{y; |y| > R_c\}.$$

The function ψ may be defined by

$$\psi(x) = A_c \exp(-(-\lambda(\mu) - c)^{1/2}|x|) \quad (x \in \mathbb{R}^N).$$

Here, the constant A_c may be chosen such that

$$(4.2) \quad \psi(x) \geq u_\mu(x) \text{ holds for almost all } x \in \{y; |y| \leq R_c\}.$$

Then it follows that $\psi \in H^1$ and that

$$(4.3) \quad \int \nabla \psi \nabla v \, dx \geq (\lambda(\mu) + c) \int \psi v \, dx$$

holds for all nonnegative functions $v \in H^1$.

Inequality (4.2) shows that $(u_\mu - \psi)_+$ is a nonnegative function on H^1 satisfying $(u_\mu - \psi)_+(x) = 0$ for almost all $x \in \{y; |y| \leq R_c\}$. Hence, we obtain from (3.4), (4.1) and (4.3) that

$$\begin{aligned} \|\nabla(u_\mu - \psi)_+\|_2^2 &= \int \nabla(u_\mu - \psi) \nabla(u_\mu - \psi)_+ \, dx \\ &\leq \lambda(\mu) \int u_\mu (u_\mu - \psi)_+ \, dx + c \int u_\mu (u_\mu - \psi)_+ \, dx \\ &\quad - (\lambda(\mu) + c) \int \psi (u_\mu - \psi)_+ \, dx \\ &= (\lambda(\mu) + c) \|(u_\mu - \psi)_+\|_2^2 \leq 0 \end{aligned}$$

and consequently that $u_\mu \leq \psi$. □

Lemma 4.2. *Let q and r satisfy the assumptions (A)–(D) and suppose that $\sigma_2 \leq \sigma_1$. Then $\lambda(\mu) < 0$ holds for all $\mu \in (0, \mu_a)$.*

PROOF: Since $\xi(u_\mu) < 0$, we see that

$$\int r |u_\mu|^{2+\sigma_2} \, dx < -((2 + \sigma_2)/2) \|\nabla u_\mu\|_2^2 + ((2 + \sigma_2)/(2 + \sigma_1)) \int q |u_\mu|^{2+\sigma_1} \, dx$$

and that

$$\lambda(\mu) < \|u_\mu\|_2^{-2} \left(-(\sigma_2/2) \|\nabla u_\mu\|_2^2 + ((\sigma_2 - \sigma_1)/(2 + \sigma_1)) \int q |u_\mu|^{2+\sigma_1} \, dx \right).$$

Then using the fact that

$$\int q |u_\mu|^{2+\sigma_1} \, dx > -(2 + \sigma_1) \xi(u_\mu) > 0,$$

we obtain the assertion. □

Now, we consider the case that $\sigma_1 < \sigma_2$. Since $I(\cdot)$ is a monotone decreasing function on $[0, \mu_a)$, we can find a measurable subset \mathcal{M} of $[0, \mu_a)$ such that $[0, \mu_a) \setminus \mathcal{M}$ has measure zero and $I(\cdot)$ is differentiable on \mathcal{M} (see [4, Theorem 17.12]). Then, we see that

$$(4.4) \quad I'(\mu) \leq 0 \text{ holds for all } \mu \in \mathcal{M}.$$

Lemma 4.3. *The function $I(\cdot)$ is Lipschitz continuous on $[0, \mu_a)$ and for all $\mu \in \mathcal{M}$ we have $I'(\mu) \geq \mu^{-1} \|u_\mu\|_2^2 \lambda(\mu)$.*

PROOF: Let $0 \leq \nu < \mu < \mu_a$. Then, we obtain

$$I(\nu) \leq \xi((\nu/\mu)u_\mu)$$

and therefore that

$$(4.5) \quad \begin{aligned} I(\nu) - I(\mu) &\leq \frac{1}{2}((\nu/\mu)^2 - 1) \int |\nabla u_\mu|^2 dx \\ &\quad - (2 + \sigma_1)^{-1}((\nu/\mu)^{2+\sigma_1} - 1) \int q|u_\mu|^{2+\sigma_1} dx \\ &\quad + (2 + \sigma_2)^{-1}((\nu/\mu)^{2+\sigma_2} - 1) \int r|u_\mu|^{2+\sigma_2} dx. \end{aligned}$$

Thus, (4.5) implies for $\mu \in \mathcal{M}$: $I'(\mu) \geq \mu^{-1} \|u_\mu\|_2^2 \lambda(\mu)$. Moreover, we obtain

$$\begin{aligned} |I(\mu) - I(\nu)| |\mu - \nu|^{-1} &= (I(\nu) - I(\mu))(\mu - \nu)^{-1} \\ &\leq (2 + \sigma_1)^{-1} (1 - (\nu/\mu)^{2+\sigma_1}) (\mu - \nu)^{-1} \int q_+ |u_\mu|^{2+\sigma_1} dx \\ &\leq (1 - (\nu/\mu)) (\mu - \nu)^{-1} \int q_+ |u_\mu|^{2+\sigma_1} dx \\ &= \mu^{-1} \int q_+ |u_\mu|^{2+\sigma_1} dx. \end{aligned}$$

Hence, Lemma 3.1 and Proposition 3.7 show that

$$|I(\mu) - I(\nu)| \leq C(\mu^{1+\alpha} + \mu^{1+\beta}) |\mu - \nu|.$$

□

Lemma 4.4. *There exists a monotone decreasing sequence $(\mu_n) \subset (0, \mu_a)$ such that $\lim_{n \rightarrow \infty} \mu_n = 0$ and $\lambda(\mu_n) < 0$ holds for all n .*

PROOF: Suppose that $\lambda(\mu) \geq 0$ holds for all $\mu \in (0, \mu_a)$. Then, according to Lemma 3.6, we see that $\lambda(\mu) = 0$ holds for all $\mu \in (0, \mu_a)$. Furthermore, (4.4) and Lemma 4.3 would imply that $I'(\mu) = 0$ for all $\mu \in \mathcal{M}$ and consequently that $I(\cdot)$ is constant on $[0, \mu_a)$ (see [4, Theorem 18.15]). In particular, we would obtain that

$$0 = I(0) = I(\min((\mu_a/2), 1)) < 0.$$

Hence, there exists a constant $\mu_1 \in (0, \mu_a)$ such that $\lambda(\mu_1) < 0$. Now, repeating this procedure, we can find a $\mu_2 \in (0, \min(\mu_1, 1/2))$ such that $\lambda(\mu_2) < 0$. Moreover, by induction we can show that for each n there is a constant $\mu_n \in (0, \min(\mu_{n-1}, 1/n))$ so that $\lambda(\mu_n) < 0$. □

Finally, we see that Lemma 4.1 and Lemma 4.2 imply Theorem 1.5 and that Theorem 1.6 is obtained by Lemma 4.1 and Lemma 4.4.

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