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On a class of commutative groupoids determined by their associativity triples

Aleš Drápal

Abstract. Let $G = G(\cdot)$ be a commutative groupoid such that $\{(a,b,c) \in G^3; a \cdot bc \neq ab \cdot c\} = \{(a,b,c) \in G^3; a = b \neq c \text{ or } a \neq b = c\}$. Then $G$ is determined uniquely up to isomorphism and if it is finite, then $\text{card}(G) = 2^i$ for an integer $i \geq 0$.

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For a groupoid $G = G(\cdot)$ denote by $\text{Ns}(G)$ the set of its non-associative triples, i.e. $\text{Ns}(G) = \{(a,b,c) \in G^3; a \cdot bc \neq ab \cdot c\}$. If $\mathcal{V}$ is a variety of groupoids and $S$ a non-empty set, then it can be a non-trivial problem to determine all such $N \subseteq S^3$ that $N = \text{Ns}(G)$ for a groupoid $G = S(\cdot) \in \mathcal{V}$. For example, it is known [1], [2] that $\text{Ns}(G) \neq \{(a,a,a); a \in G\}$ for any non-empty groupoid $G$.

In the present short note we investigate the case when $\mathcal{V}$ is the variety of the commutative groupoids and $\text{Ns}(G) = \{(a,b,c) \in G^3; a = b \neq c \text{ or } a \neq b = c\}$. We shall show that all such non-trivial groupoids can be obtained by a slight modification of a 2-elementary Abelian group and that these groupoids are determined up to isomorphism by $\text{card}(G)$. Moreover, whenever $G$ is finite and non-trivial, then $\text{card}(G) = 2^i$ for an integer $i \geq 1$.

Note that $a \cdot ba = ab \cdot a$ for any $a,b \in G$ whenever $G$ is a commutative groupoid. The set $\{(a,b,c) \in G^3; a = b \neq c \text{ or } a \neq b = c\}$ thus covers all $(a,b,c) \in G^3$ such that $\text{card}\{a,b,c\} \leq 2$ and $a \cdot bc \neq ab \cdot c$ can occur.

Theorem 1. For an Abelian group $G(+)$ and each $0 \neq e \in G$ define on the set $G$ a commutative groupoid $G_e$ by $0 \cdot 0 = e$, $a \cdot b = a + b$ and $a \cdot 0 = 0 \cdot a = 0$ for any $a,b \in G \setminus \{0\}$. If $G(\cdot)$ is 2-elementary, then $\text{Ns}(G_e) = \{(a,b,c) \in G^3; a = b \neq c \text{ or } a \neq b = c\}$. Conversely, if $G(\cdot)$ is a commutative groupoid where $a \cdot bc \neq ab \cdot c$ if and only if $a = b \neq c$ or $a \neq b = c$, and $\text{card}(G) > 1$, then there exist a 2-elementary Abelian group $G(+)$ and an element $0 \neq e \in G$ such that $G(\cdot) = G_e$. Moreover, $G_e$ is isomorphic to $G_f$ for any choice of $e,f \in G$, $e \neq 0 \neq f$.

Proof: Only the converse part of the theorem requires a proof. Let us hence assume that $G(\cdot)$ is a commutative groupoid, $\text{card}(G) > 1$ and $\text{Ns}(G(\cdot)) = \{(a,b,c) \in G^3; a = b \neq c \text{ or } a \neq b = c\}$. As $G$ is commutative, we have

(1) $a \cdot ba = ab \cdot a$ for any $a,b \in G$. 


Let \( a = bc \), where \( a, b, c \in G \) are pair-wise distinct. If \( c \neq ab \), then \( aa \cdot b = (bc \cdot a) \cdot b = (b \cdot ca) \cdot b = b(ca \cdot b) = b(c \cdot ab) = bc \cdot ab = a \cdot ab \). Hence \( c = ab \) and we have

(2) If \( a = bc \), \( b \neq a \neq c \) and \( b \neq c \), then \( b = ac \) and \( c = ab \).

Further, we shall prove

(3) If \( a = bc \), \( b \neq a \neq c \) and \( b \neq c \), then \( a^2 = b^2 = c^2 \) and \( a^2 \notin \{a, b, c\} \).

To see this, observe that \( c^2 = ab \cdot c = a \cdot bc = a^2 \) by (2) and that \( a^2 = a \) implies \( a \cdot bb = a \cdot aa = a^3 = a \cdot b = ab \cdot b \). Hence we have

(4) \( a = a^2 \) or \( a^2 = a^3 \) or \( a = a^3 \) for any \( a \in G \).

Let \( a, b, c \in G \) be again pair-wise distinct and \( a = bc \). Then \( a \neq a^2 \) by (3), and \( a^3 = a \) implies \( a \cdot bb = a \cdot aa = a^3 = a = c \cdot b = ab \cdot b \). Hence we have

(5) If \( a = bc \), \( b \neq a \neq c \) and \( b \neq c \), then \( a \cdot a^2 = b \cdot a^2 = c \cdot a^2 = a^2 = b^2 = c^2 \).

We shall now order the set \( G \) by \( a < b \) iff \( ab = b \) and \( a \neq b \). From \( a < b \) and \( b < a \) it follows \( b = ab = ba = a \) and from \( a < b < c \) we obtain \( ac = a \cdot bc = ab \cdot c = bc = c \). Therefore \( < \) really is a (sharp) ordering of \( G \).

Let again \( a, b, c \in G \) be pair-wise distinct and with \( a = bc \). If \( e < a \), then \( ec = e \cdot ab = ea \cdot b = ab = c \). Consequently, we have

(6) Let \( a = bc \), \( b \neq a \neq c \) and \( b \neq c \). If \( e < a \), then \( e < b \) and \( e < c \).

Conversely, suppose that \( a < e \). Then \( b \neq e \neq c \), \( eb = ea \cdot b = e \cdot ab = ec \) and \( eb \cdot c = e \cdot bc = ea = e \). From \( eb = c \) it follows \( ec = c \), \( e < c \) and by (2) and (6) \( e < a \). Therefore \( eb \neq c \). If \( eb, c, e \) are pair-wise distinct, then \( a < e \) implies by (6) that \( a < c \), a contradiction. It follows \( eb = e \) and we obtain

(7) Let \( a = bc \), \( b \neq a \neq c \) and \( b \neq c \). If \( a < e \), then \( b < e \) and \( c < e \).

For \( a, b \in G \) put \( (a, b) \in r \) iff \( a \neq b \) and \( a \neq ab \neq b \), and denote by \( \sim \) the least equivalence containing the relation \( r \). From (6) and (7) we get by induction immediately

(8) Let \( a, b, e \in G \) and let \( a \sim b \). Then \( a < e \) iff \( b < e \), and \( e < a \) iff \( e < b \).

Denote by \( E \) the set of equivalence classes of \( \sim \). By the definitions of \( \sim \) and \( < \) we have either \( a \sim b \), or \( a < b \), or \( b < a \) for any \( a, b \in G \). Hence it follows from (8) that \( < \) induces a linear ordering of \( E \). Suppose that \( (E, <) \) has no maximum element. Then for \( a \in G \) we can choose \( b \in G \) with \( a < b, a^2 < b \). Then \( b \cdot aa = b \cdot a^2 = b = ba = ba \cdot a \), a contradiction.

Let \( U \in E \) be the maximum element of \( (E, <) \) and suppose that \( a, b \in U, a \neq b \). Then \( a \neq ab \neq b \), and we obtain \( a^2 > a \) by (3) and (5). This is a contradiction, and hence \( U \) contains exactly one element, say \( u \).

For \( u \neq a \in G \) we have \( ua = u \), and thus \( u = ua \cdot a \neq u \cdot a^2 \) provides \( a^2 = u \). Therefore \( a < b < u \) would imply \( a \cdot bb = au = u = bb = ab \cdot b \), which contradicts our hypothesis. It follows that the equivalence \( \sim \) has exactly two classes and by (7) we have \( ab \notin \{a, b, u\} \) for any \( a, b \in G, a \neq b, a \neq u \neq b \). Moreover, \( u^2 = aa \cdot u \neq a \cdot au = u \).
Put now $u = 0$ and define $G(\cdot)$ by $a + 0 = 0 + a = a$ for any $a \in G$ and $a + b = ab$ for $a, b \in G$, $a \neq 0 \neq b$. Clearly, $a + (b + c) = (a + b) + c$ whenever $0 \notin \{a, b, c\}$, and by (2) also when $a = b$ or $b = c$. Similarly, $a + (b + ab) = a + a = ab + ab = ab + (a + b)$ for $a, b \in G$, $a \neq b$, $a \neq 0 \neq b$. Finally, $a + (b + c) = a + bc = a \cdot bc = ab \cdot c = ab + c = (a + b) + c$ when $a, b, c \in G$ are pair-wise distinct and $c \neq ab$. It follows that $G(\cdot)$ is a 2-elementary Abelian group and we see that $G(\cdot) = G_{u^2}$. □

References


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