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Köthe dual of Banach sequence spaces
\(\ell_p[X]\) \((1 \leq p < \infty)\) and Grothendieck space

Wu Congxin, Bu Qingying

Abstract. In this paper, we show the representation of Köthe dual of Banach sequence spaces \(\ell_p[X]\) \((1 \leq p < \infty)\) and give a characterization of that the spaces \(\ell_p[X]\) \((1 < p < \infty)\) are Grothendieck spaces.

Keywords: vector-valued sequence space; Köthe dual; GAK-space; Grothendieck space
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Let \(X\) be a Banach space and \(X^*\) its topological dual, and let \(B_X\) denote the closed unit ball of \(X\). For \(1 \leq p < \infty\), let

\[\ell_p(X) = \{\overline{x} = (x_j) \in X^N : \|\overline{x}\|_{\ell_p} = \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{1/p} < \infty\},\]

\[\ell_p[X] = \{\overline{x} = (x_j) \in X^N : \text{for each } f \in X^*, \sum_{i \geq 1} |f(x_i)|^p < \infty\}.\]

And for each \(\overline{x} \in \ell_p[X]\), let

\[\|\overline{x}\|_{(\ell_p)} = \sup\left\{\left(\sum_{i \geq 1} |f(x_i)|^p\right)^{1/p} : f \in B_{X^*}\right\}.\]

Then \((\ell_p(X), \| \cdot \|_{\ell_p})\) and \((\ell_p[X], \| \cdot \|_{(\ell_p)})\) are Banach spaces (see [1], [2], [3]). For \(\overline{x} \in X^N\), let

\[\overline{x} (i \leq n) = (x_1, \ldots, x_n, 0, 0, \ldots),\]

\[\overline{x} (i > n) = (0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots).\]

And let

\[\ell_p[X]_r = \{\overline{x} \in \ell_p[X] : \lim_n \|\overline{x} (i > n)\|_{(\ell_p)} = 0\}.\]

If \(\ell_p[X]_r = \ell_p[X]\), then \(\ell_p[X]\) is said to be a GAK-space [4].

For a vector-valued sequence space \(S(X)\) from \(X\), define its Köthe dual with respect to the dual pair \((X, X^*)\) (see [4]) as follows:

\[S(X)\times |_{(X, X^*)} = \{\overline{f} = (f_j) \in X^N : \text{for each } \overline{x} = (x_j) \in S(X), \sum_{i \geq 1} |f_i(x_i)| < \infty\}.\]

We denote \(S(X)\times |_{(X, X^*)}\) by \(S(X)\times\) simply if the meaning is clear from the context.

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Lemma 1. For $1 \leq p < \infty$, $(\ell_p[X],r) = \ell_p[X]$.  

Proof: It is easy to see that $\ell_p[X] \subseteq (\ell_p[X])$. So we only need to prove that $(\ell_p[X],r) \subseteq \ell_p[X]$.  

For $x = (x_j) \in \ell_p[X]$ and $t = (t_j) \in c_0$, let $t \bar{x} = (t_j x_j)$. Then $\|t \bar{x}\| (i > n) \leq \|x\| \sup_{i > n} \|t_i\|$ implies that $t \bar{x} \in \ell_p[X]$. So for $f = (f_j) \in (\ell_p[X])$, we have  

$$\sum_{i \geq 1} |f_i(t_i x_i)| < \infty.$$  

It follows from the fact that $t \in c_0$ was taken arbitrary that  

$$\sum_{i \geq 1} |f_i(x_i)| < \infty.$$  

Thus, $f \in \ell_p[X]$ and the proof is completed.  

Lemma 2. (1) For $1 \leq p < \infty$, $\ell_p[X] \subseteq (\ell_p[X], \| \cdot \|_{(\ell_p)})$ and $(\ell_p[X], r) = (\ell_p[X], \| \cdot \|_{(\ell_p)})$.  

(2) Let $\| \cdot \|_{(\ell_p)}$ denote the dual norm of $\| \cdot \|_{(\ell_p)}$ on the dual space $(\ell_p[X], \| \cdot \|_{(\ell_p)})$. Then for each $f \in \ell_p[X]$, we have  

$$\|f\|_{(\ell_p)} = \sup \{|f(x)| : f \in \ell_p[X], \|f\|_{(\ell_p)} \leq 1\},$$  

where $\langle f, \bar{x} \rangle = \sum_{i \geq 1} f_i(x_i)$.  

Proof: See Theorem 2.3 in [3].  

Lemma 3. Every weak* unconditionally Cauchy series in $X^*$ is weak unconditionally Cauchy series.  

Proof: See the proof of p. 49, Corollary 11 in [5].  

Lemma 4. For $1 \leq p < \infty$,  

$$\ell_p[X^*] = \left\{ f = (f_j) \in X^* : \text{for each } x \in X, \sum_{i \geq 1} |f_i(x)| < \infty \right\}.$$  

Proof: Let  

$$\Delta = \left\{ f = (f_j) \in X^* : \text{for each } x \in X, \sum_{i \geq 1} |f_i(x)| < \infty \right\}.$$  

By definition, we only need to prove that $\Delta \subseteq \ell_p[X^*]$.  

Let $f \in \Delta$ and $t_j \in \ell_q(1/p + 1/q = 1)$. Then $\sum_{i \geq 1} |f_i(t_i x)| < \infty$ for each $x \in X$. So the series $\sum_{j} t_j f_j$ is weak* unconditionally Cauchy in $X^*$ and hence, it is weak unconditionally Cauchy by Lemma 3. That is, $\sum_{i \geq 1} |F(t_i f_i)| < \infty$ for each $F \in X^{**}$. Since $(t_j)$ is arbitrary in $\ell_q$, $\sum_{i \geq 1} |F(f_i)| < \infty$ and $f = (f_j) \in \ell_p[X^*].$ The proof is completed.
Lemma 5 (the principle of local reflexivity, [6]). Let $X$ be a normed space and $Z^{**}$ a finite dimensional subspace of $X^{**}$. For $\{F_i\}_1^n \subseteq Z^{**}$, $\{f_i\}_1^n \subseteq X^*$ and $\varepsilon > 0$, there exists a linear map $T : Z^{**} \to X$ such that $\|T\| \leq 1$ and

$$|f_i(TF_i) - F_i(f_i)| < \varepsilon, \quad i = 1, 2, \ldots, n.$$  

Proposition 6. $\ell_p[X^{**}]_1 \cap (X^*,X^*) = \ell_p[X]_1 \cap (X,X^*)$ (1 $\leq p < \infty$).

Proof: It is easy to see that $\ell_p[X]_1 \subseteq \ell_p[X^{**}]$ implies that

$$\ell_p[X^{**}]_1 \cap (X^*,X^*) \subseteq \ell_p[X]_1 \cap (X,X^*).$$

So we only need to prove that

$$\ell_p[X]_1 \cap (X,X^*) \subseteq \ell_p[X^{**}]_1 \cap (X^*,X^*).$$

Let $\bar{f} = (f_j) \in \ell_p[X]_1 \cap (X,X^*)$ and $\bar{F} = (F_j) \in \ell_p[X^{**}]$. For a fixed $n \in \mathbb{N}$, by Lemma 5, there exists a linear map $T_n : \text{span} \{F_i\}_1^n \to X$ such that $\|T_n\| \leq 1$ and

$$|F_i(f_i)| \leq |f_i(T_nF_i)| + 1/n, \quad i = 1, 2, \ldots, n.$$  

Now we prove that $\{(T_nF_1, \ldots, T_nF_n, 0, 0, \ldots)\}_{n=1}^\infty$ is a bounded subset of $\ell_p[X]$. By Theorem 1.5 in [2], we have

$$\|(T_nF_1, \ldots, T_nF_n, 0, 0, \ldots)\|_{\ell_p(|X^*,X^*)}) = \sup\left\{\left\|\sum_{i=1}^n s_iT_nF_i\right\| : s = (s_j) \in B_{\ell_q}\right\} (1/p + 1/q) = 1$$

$$\leq \sup\left\{\|T_n\|\left\|\sum_{i=1}^n s_iF_i\right\| : s \in B_{\ell_q}\right\}$$

$$\leq \sup\left\{\left\|\sum_{i=1}^\infty s_iF_i\right\| : s \in B_{\ell_q}\right\} = \|\bar{F}\|_{\ell_p(|X^*,X^*)}).$$

So $\{(T_nF_1, \ldots, T_nF_n, 0, 0, \ldots)\}_{n=1}^\infty$ is a bounded subset of $\ell_p[X]$ and hence, $\sigma(\ell_p[X], \ell_p[X]_1 \cap (X,X^*))$-bounded. Thus, we have

$$\sum_{i=1}^n |F_i(f_i)| \leq \sum_{i=1}^n |f_i(T_nF_i)| + 1 \leq \sup_{n \geq 1}\left\{\sum_{i=1}^n |f_i(T_nF_i)|\right\} + 1.$$  

Because $n \in \mathbb{N}$ is arbitrary, it follows that

$$\sum_{i=1}^\infty |F_i(f_i)| < \infty.$$  

So we prove that $\bar{f} = (f_j) \in \ell_p[X^{**}]_1 \cap (X^{**},X^*)$ and this completes the proof. \hfill $\square$
Proposition 7. \((\ell_p[X]^\times |_ {(X,X^*)})^\times |_ {(X^*,X^{**})} = \ell_p[X^{**}] (1 \leq p < \infty)\).

Proof: By Proposition 6, it is easy to see that

\[
\ell_p[X^{**}] \subseteq (\ell_p[X^{**}]^\times |_ {(X^{**},X^*)})^\times |_ {(X^*,X^{**})} = (\ell_p[X]^\times |_ {(X,X^*)})^\times |_ {(X^*,X^{**})}.
\]

So we only need to prove that

\[
(\ell_p[X^{**}]^\times |_ {(X^{**},X^*)})^\times |_ {(X^*,X^{**})} \subseteq \ell_p[X^{**}].
\]

Let \(F = (F_j) \in (\ell_p[X^{**}]^\times |_ {(X^{**},X^*)})^\times |_ {(X^*,X^{**})}\). Since \(f \in X^*\) and \(t = (t_j) \in \ell_q (1/p + 1/q = 1)\) implies that \((t_j f) \in \ell_p[X^{**}]^\times |_ {(X^{**},X^*)}, \sum_{i \geq 1} |F_i(t_i f)| < \infty\). Thus, \(\sum_{i \geq 1} F_i(f)^p < \infty\) and hence, \(F \in \ell_p[X^{**}]\) by Lemma 4. The proof is completed.

Theorem 8. For \(1 \leq p < \infty\), \(\ell_p \overset{\sim}{\otimes} X\), the injective tensor product of \(\ell_p\) and \(X\), is isometrically isomorphic to the space \((\ell_p[X]^r, \| \cdot \|_{(\ell_p)})\).

Proof: For each \(u = \sum_{i=1}^n \frac{t(i)}{q} x_i \in \ell_p \otimes X\) \((t(i) \in \ell_p, x_i \in X)\), define \(\overline{u} = (\sum_{i=1}^n \frac{t(i)}{q} x_i, \sum_{i=1}^n \frac{t(i)}{q} x_i, \ldots)\). Then

\[
\|\overline{u}\|_{(\ell_p)} = \sup \left\{ \sum_{k \geq 1} s_k f \left( \sum_{i=1}^n t^{(i)} k x_i \right) : f \in B_{X^*}, s \in B_{\ell_q} \right\}
\]

\[
= \sup \left\{ \sum_{i=1}^n f(x_i) \langle t^{(i)}, s \rangle : f \in B_{X^*}, s \in B_{\ell_q} \right\}
\]

\[
= \lambda(u) \quad \text{(see [7, p. 223])} \quad (1/p + 1/q = 1).
\]

Let \(M = \sup_{1 \leq i \leq n} \|x_i\|\). It follows from the above equality that

\[
\|\overline{u} (j > k)\|_{(\ell_p)} = \sup \left\{ \sum_{i=1}^n f(x_i) \langle t(i), s (j > k) \rangle : f \in B_{X^*}, s \in B_{\ell_q} \right\}
\]

\[
\leq M \sup \left\{ \sum_{i=1}^n |(t^{(i)}, s (j > k))| : s \in B_{\ell_q} \right\}.
\]

Since \(B_{\ell_q}\) is weak* compact, Theorem 6.11 in [8] implies that

\[
\lim_k \|\overline{u} (j > k)\|_{(\ell_p)} = 0.
\]

So, \(\overline{u} \in \ell_p[X]^r\), and we can define a map \(\varphi : \ell_p \otimes X \to \ell_p[X]^r\) by \(\varphi(u) = \overline{u}\). It is easy to see that \(\varphi\) is a linear isometrically isomorphic map from \(\ell_p \otimes X\) to \(\ell_p[X]^r\). Next, we only need to prove that \(\varphi\) is surjective.
For $\varphi = (x_1, \ldots, x_n, 0, 0, \ldots)$, if we let $u = \sum_{i=1}^{n} e_i \otimes x_i$ (where $e_i = (0, \ldots, 0, 1^{(i)}, 0, 0, \ldots)$), then $\varphi = \varphi(u)$. Notice that $\lim_n \varphi(j \leq n) = \varphi$ for each $\varphi \in \ell_p[X]_r$. So $\varphi$ is surjective and the proof is completed. □

For two Banach spaces $X$ and $Y$, let $B^\wedge(X,Y)$, $I(X,Y)$ and $N(X,Y)$ denote the class of integral bilinear functionals on $X \times Y$, the class of integral operators from $X$ to $Y$ and the class of nuclear operators from $X$ to $Y$ respectively (see p. 232 and p. 170 in [7]).

**Theorem 9.** Let $1 \leq p < \infty$ and $1/p + 1/q = 1$. Then $\overline{f} = (f_j) \in \ell_p[X]^\times \mid_{(X,X^*)}$ if and only if there exist an $r = (r_j) \in \ell_1$ a bounded sequence $\{s^{(n)}\}_{n=1}^\infty$ of $\ell_q$ and a bounded sequence $\{h_n\}_{n=1}^\infty$ of $X^*$ such that

$$f_i = \sum_{n \geq 1} r_n s_i^{(n)} h_n, \quad i = 1, 2, \ldots.$$  

**Proof:** Necessity. Let $\overline{f} = (f_j) \in \ell_p[X]^\times$. By Lemma 1 and Lemma 2, $\overline{f} \in (\ell_p[X], \| \cdot \|_{(\ell_p)}^\times)$. So Theorem 8 implies that there is an $\psi^* \in (\ell_p \overset{\vee}{\times} X)^*$ corresponding to $\overline{f}$. By Definition 6 in [7, p. 232], there is an $\psi \in B^\wedge(\ell_p, X)$ corresponding to $\psi^*$. Furthermore, by Corollary 12 in [7, p. 237], there exists a $T_\psi \in I(\ell_p, X^*)$ corresponding to $\psi$. Since Corollary 10 in [7, p. 235] and Theorem 6 in [7, p.248] guarantee that $I(\ell_p, X^*) = N(\ell_p, X^*)$, there are an $r = (r_j) \in \ell_1$, a bounded sequence $\{s^{(n)}\}_{n=1}^\infty$ of $\ell_q$ and a bounded sequence $\{h_n\}_{n=1}^\infty$ of $X^*$ such that

$$T_\psi(t) = \sum_{n \geq 1} r_n \langle t, s^{(n)} \rangle h_n, \quad \text{for } t \in \ell_p.$$  

Now for each $i \geq 1$ and each $x \in X$, by the above corresponding relations, we have

$$T_\psi(e_i)(x) = \psi(e_i, x) = \psi^*(e_i \otimes x) = \langle \varphi(e_i \otimes x), \overline{f} \rangle = f_i(x).$$  

Thus

$$f_i = T_\psi(e_i) = \sum_{n \geq 1} r_n s_i^{(n)} h_n, \quad i = 1, 2, \ldots.$$  

Sufficiency. Let $M = \sup_{n \geq 1} \| s^{(n)} \|_q$ and $N = \sup_{n \geq 1} \| h_n \|$. Then, for each $x = (x_j) \in \ell_p[X]$, we have

$$\sum_{i \geq 1} |s_i^{(n)} h_n(x_i)| \leq MN \| \varphi \|_{(\ell_p)}, \quad \text{for } n \geq 1.$$  

And so

$$\sum_{i \geq 1} |f_i(x_i)| \leq \sum_{n \geq 1} |r_n| \sum_{i \geq 1} |s_i^{(n)} h_n(x_i)| < \infty.$$  

Therefore, $\overline{f} \in \ell_p[X]^\times$ and the proof is completed. □
Theorem 10. For $1 < p < \infty$, $(\ell_p[X]^\times, \| \cdot \|^*_p)$ is a GAK-space.

Proof: Let $\mathcal{T} = (f_j) \in \ell_p[X]^\times$. Then by Theorem 9, there exist an $r = (r_j) \in \ell_1$, a bounded sequence $\{s^{(n)}\}_1^n$ of $\ell_q$ and a bounded sequence $\{h_n\}_1^\infty$ of $X^*$ such that

$$f_i = \sum_{n \geq 1} r_n s^{(n)}_i h_n, \quad i = 1, 2, \ldots.$$  

Without loss of generality, we can assume that $\|s^{(n)}\|_q \leq 1$ and $\|h_n\| \leq 1$ for $n \geq 1$. Thus, for $x \in \ell_p[X]$ with $\|x\|_p \leq 1$, we have

$$\sum_{i \geq 1} |s^{(n)}_i h_n(x_i)| \leq \|x\|_p \leq 1 \quad \text{for } n \geq 1.$$  

So

$$\left\{ \left( \sum_{i \geq 1} |s^{(n)}_i h_n(x_i)| \right)_{n \geq 1} : \|x\|_p \leq 1 \right\} \subseteq B_{\ell_\infty}.$$  

Let $\varepsilon > 0$. Then $B_{\ell_\infty}$ is weak* compact implies that there exists an $n_0 \in \mathbb{N}$ such that

$$\sum_{n > n_0} |r_n| \sum_{i \geq 1} |s^{(n)}_i h_n(x_i)| < \varepsilon/2, \quad x \in \ell_p[X], \quad \|x\|_p \leq 1.$$  

Since $B_{\ell_p}$ is weakly compact set and

$$\left\{ (h_n(x_i))_{i \geq 1} : x \in \ell_p[X], \quad \|x\|_p \leq 1, \quad n \geq 1 \right\} \subseteq B_{\ell_p},$$

there is a $k_0 \in \mathbb{N}$ such that for each $k > k_0$,

$$\sum_{i > k} |s^{(n)}_i h_n(x_i)| < \varepsilon/2 \|r\|_1$$

for $x \in \ell_p[X]$ with $\|x\|_p \leq 1$ and $n = 1, 2, \ldots, n_0$. Thus, for each $x \in \ell_p[X]$ with $\|x\|_p \leq 1$ and each $k > k_0$, we have

$$\sum_{i > k} |f_i(x_i)| \leq \sum_{n = 1}^{n_0} |r_n| \sum_{i > k} |s^{(n)}_i h_n(x_i)| + \sum_{n > n_0} |r_n| \sum_{i > k} |s^{(n)}_i h_n(x_i)|$$

$$\leq \left( \sum_{n = 1}^{\infty} |r_n| \right) \varepsilon/2 \|r\|_1 + \sum_{n > n_0} |r_n| \sum_{i \geq 1} |s^{(n)}_i h_n(x_i)| < \varepsilon.$$  

So for $k > k_0$,

$$\|\mathcal{T} (j > k)\|^*_p \leq \sup \left\{ |\langle x, \mathcal{T} (j > k) \rangle| : x \in \ell_p[X], \quad \|x\|_p \leq 1 \right\}$$

$$= \sup \left\{ |\sum_{i > k} f_i(x_i)| : \|x\|_p \leq 1 \right\} < \varepsilon.$$
Therefore, \( \lim_k \| \mathbf{f} \|_{(\ell_p^*)} = 0 \) and \( \mathbf{f} \in (\ell_p[X], \| \cdot \|_{(\ell_p^*)})_r \).

For \( 1 < p < \infty \), by Theorem 10 and [4, Proposition 4.9], we have

\[
(\ell_p[X]^\infty \mid (X,X^*)^\infty \mid (X^*,X^{**}) = (\ell_p[X]^\infty \mid (X,X^*)^\infty \mid (X^*,X^{**})) - \lim_n f(n) = 0
\]

is equivalent to

(a) \( \sigma(X^*, X^{**}) - \lim_n f_i(n) = 0 \) for \( i \geq 1 \); and

(b) \( \sup_{n \geq 1} \| f(n) \|_{(\ell_p^*)} < \infty \)

if and only if \( ((\ell_p[X]^\infty \mid (X,X^*)^\infty \mid (X^*,X^{**}), \| \cdot \|_{(\ell_p^*)}) \) is a GAK-space.

**Proposition 12.** Let \( \mathbf{f}(n) \in (\ell_p[X])^* \) (\( 1 \leq p < \infty \)). Then

\[
\sigma((\ell_p[X]^r)^*, \ell_p[X^r]) - \lim_n \mathbf{f}(n) = 0
\]

if and only if \( \sigma(X^*, X) - \lim_n f_i(n) = 0 \) for \( i \geq 1 \) and \( \sup_{n \geq 1} \| f(n) \|_{(\ell_p^*)} < \infty \).

We say a Banach space \( X \) to be a Grothendieck space if every weak* null sequence on \( X^* \) is weak null sequence (see [7, p. 179]). Leonard [1] has proved that \( \ell_p(X) \) (\( 1 < p < \infty \)) is a Grothendieck space if and only if \( X \) is a Grothendieck space. Now we have

**Theorem 13.** For \( 1 < p < \infty \). The Banach space \( (\ell_p[X]^r, \| \cdot \|_{(\ell_p)}) \) is a Grothendieck space if and only if

(i) \( X \) is a Grothendieck space; and

(ii) \( (\ell_p[X^{**}], \| \cdot \|_{(\ell_p)}) \) is a GAK-space.

**Proof:** Sufficiency. By (ii), \( (\ell_p[X], \| \cdot \|_{(\ell_p)}) \) is a GAK-space, i.e. \( \ell_p[X^r] = \ell_p[X] \).

Let \( \mathbf{f}(n) \in (\ell_p[X], \| \cdot \|_{(\ell_p)})^* \) such that

\[
\sigma(\ell_p[X]^*, \ell_p[X]) - \lim_n \mathbf{f}(n) = 0.
\]
By Proposition 12, we have
\[ \sigma(X^*, X) - \lim_{n} f^{(n)}_i = 0, \quad i = 1, 2, \ldots \]
and
\[ \sup_{n \geq 1} \|f^{(n)}_i\|_{(\ell_p)} < \infty. \]
By (i), we have
\[ \sigma(X^*, X^{**}) - \lim_{n} f^{(n)}_i = 0, \quad i = 1, 2, \ldots, \]
By (ii) and Propositions 2, 6, 7, the space \((\ell_p[X]^* \times (X^*, X^{**}), \|\cdot\|_{(\ell_p)})\) is a GAK-space. So Proposition 11 guarantees that
\[ \sigma(\ell_p[X]^*, (\ell_p[X]^*)^\times) - \lim_{n} f^{(n)} = 0. \]
It follows from (*) that
\[ \sigma(\ell_p[X]^*, \ell_p[X]^{**}) - \lim_{n} f^{(n)} = 0. \]
and completes the sufficiency.

Necessity. To prove (i), let \(f_n \in X^* \ (n \geq 1)\) such that
\[ \sigma(X^*, X) - \lim_{n} f_n = 0. \]
Let \(f^{(n)} = (f_n, 0, 0, \ldots)\) for \(n \geq 1\). Then \(f^{(n)} \in (\ell_p[X]_r)^*\) and
\[ \sigma((\ell_p[X]_r)^*, \ell_p[X]_r) - \lim_{n} f^{(n)} = 0. \]
So
\[ \sigma((\ell_p[X]_r)^*, (\ell_p[X]_r)^{**}) - \lim_{n} f^{(n)} = 0 \]
and hence, \(\sigma(X^*, X^{**}) - \lim_n f_n = 0\). (i) follows.

For (ii), let \(f^{(n)} \in \ell_p[X]^* \times (X^*, X^*)\) such that
\[ \sigma(X^*, X^{**}) - \lim_{n} f^{(n)}_i = 0, \quad i = 1, 2, \ldots, \]
and
\[ \sup_{n \geq 1} \|f^{(n)}_i\|_{(\ell_p)} < \infty. \]
By Lemmas 1, 2 and Proposition 12, we have
\[ \sigma((\ell_p[X]_r)^*, \ell_p[X]_r) - \lim_{n} f^{(n)} = 0. \]
And hence,
\[ \sigma((\ell_p[X]_r)^*, (\ell_p[X]_r)^{**}) - \lim_{n} f^{(n)} = 0. \]
It follows from (*) that
\[ \sigma(\ell_p[X]^* \times (X^*, X^*)^\times) - \lim_{n} f^{(n)} = 0. \]
So Propositions 6, 7, 11 imply that \((\ell_p[X]^{**}, \|\cdot\|_{(\ell_p)})\) is a GAK-space and (ii) follows.
The proof is completed.
Corollary 14. If $\ell_p[X]_r$ (1 < $p < \infty$) is a Grothendieck space, then $\ell_p[X]$ is a GAK-space.

References


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