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Abstract. Normal spaces are characterized in terms of an insertion type theorem, which implies the Katětov-Tong theorem. The proof actually provides a simple necessary and sufficient condition for the insertion of an ordered pair of lower and upper semicontinuous functions between two comparable real-valued functions. As a consequence of the latter, we obtain a characterization of completely normal spaces by real-valued functions.

Keywords: normal space, semicontinuous functions, insertion, limit functions, completely normal space

Classification: 54D15, 54C30

Urysohn’s lemma states that if $X$ is a normal space and $K$ is its closed subset included in an open subset $U$, then there is a continuous function $f$ on $X$ such that $\chi_K \leq f \leq \chi_U$. The well-known Katětov-Tong insertion theorem strengthens this characterization by replacing $\chi_K$ and $\chi_U$ by arbitrary upper and lower semicontinuous functions. Another characterization due to Urysohn asserts that each two separated $F_\sigma$-sets in $X$ have disjoint open neighbourhoods. In terms of the upper and lower limit functions, this property can be stated as follows: given an $F_\sigma$-set $A$ and a $G_\delta$-set $B$ with $(\chi_A)^* \leq \chi_B$ and $\chi_A \leq (\chi_B)^*$, there is a lower semicontinuous function $g$ on $X$ such that $\chi_A \leq g \leq g^* \leq \chi_B$.

We shall show that this is possible in the more general case in which the characteristic functions are replaced respectively by a countable supremum of upper semicontinuous functions and a countable infimum of lower semicontinuous functions. This result can indeed be claimed to be a strengthening of the Katětov-Tong theorem, because a pair of lower and upper semicontinuous functions between an upper and a lower semicontinuous function is all we need to arrive at the Katětov-Tong result, in which one has a continuous function in-between. Namely, applying the insertion procedure repeatedly (as in the Urysohn’s lemma construction) will produce the desired function. Since we will not refer to the theorem of Katětov and Tong, the present proof provides a technique of proving it, which is different from those of Katětov [8, Theorem 1], Tong [18, Theorem 2], Priestley [15, Theorem 2], Jameson [7, 12.1], and Engelking [6, 2.7.2 (c)]. (See also Michael [13, Theorem 3.1’], Blatter and Seever [5, Interposition Theorem], Lane [12, Theorem 2.1], Preiss and Vilímovský [14, 3.4], Blair [3, 3.5], Blair and Swardson [4, 1.6], Kubiak [10, 3.7],

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and Kotzé and Kubiak [9, 3.6] for generalizations of the Katětov-Tong theorem.)
In fact, the proof does slightly more than this, and gives a simple condition which
is necessary and sufficient for the insertion of an ordered pair of lower and up-
ner semicontinuous functions between two comparable real-valued functions. (Note
that such a condition for the insertion of a continuous function is given in the above
mentioned theorems of [5] (implicitly), [12], [14], and [3].) As a consequence of
the latter, we obtain a characterization of completely normal spaces in terms of
real-valued functions. By using a different method, this was proved directly in [11,
Theorem 1].

1. Preliminaries.
Let us now introduce some notation and terminology. For
$X$ a topological space,
$F(X)$ is the set of all real-valued functions on $X$ with the usual pointwise ordering.
All the ‘sup’s ($\vee$) and ‘inf’s ($\wedge$) in $F(X)$ are understood in this pointwise order.
Given $f \in F(X)$ and $t \in \mathbb{R}$, $[f < t] = \{x \in X : f(x) < t\}$ and $[f > t] = \{x \in X : f(x) > t\}$; a similar convention applies to $[f \leq t]$ and $[f \geq t]$. The characteristic
function of $A \subset X$ is denoted by $\chi_A$. The constant map in $F(X)$ whose constant
value is $t \in \mathbb{R}$ will be denoted by $t$. We denote by $LSC(X)$, $USC(X)$, and $C(X)$
the collections of all lower semicontinuous, upper semicontinuous, and continuous
members of $F(X)$. As always, $\mathbb{Q}$ is the set of all rationals and $I = [0, 1]$.

In formulating our results, we shall use the upper and lower limit functions
$a^*$ and $a_*$ of a given $a \in F(X)$, which are defined as follows: for all $x \in X$
$$a^*(x) = \wedge \{\vee a(U) : U \text{ is an open nbhd of } x\},$$
and, dually,
$$a_*(x) = \vee \{\wedge a(U) : U \text{ is an open nbhd of } x\}.$$  
In general, $a^*$ and $a_*$ are extended real-valued functions. In this paper, however,
we shall deal only with the case that $a^* \leq b$ and $a \leq b_*$, with $a, b \in F(X)$. Then
$a^*, b_* \in F(X)$, since $a \leq a^*$ and $b_* \leq b$. Also, $a^* \in USC(X)$ and $a_* \in LSC(X)$,
and $a^* = a$ iff $a \in USC(X)$, while $a_* = a$ iff $a \in LSC(X)$. Clearly, $a \leq b$ implies
$a^* \leq b^*$ and $a_* \leq b_*$. Given a family $\mathcal{A} \subset F(X)$, we write $\mathcal{A}^* = \{a^* : a \in \mathcal{A}\}$ and
$\mathcal{A}_* = \{a_* : a \in \mathcal{A}\}$. (See [2, 5.4] or [17].)

We note that these concepts are not indispensable for this paper and we use
them just for convenience. Otherwise, instead of $a^* \leq b$, for example, one has to
write $a \leq h \leq b$ with $h \in USC(X)$. Moreover, the use of them emphasizes the
parallelism between the operators of interior and closure on the power set of $X$
and the operators $(\cdot)^*$ and $(\cdot)_*$ on $F(X)$. In particular, one has $(\chi_A)_* = \chi_{\text{Int } A}$
and $(\chi_A)^* = \chi_{\overline{A}}$.[2]

2. Insertion theorems.
We begin with the following lemma which extends the so-called normalization
lemma (see Aleksandrov and Pasynkov [1, Chap. I, §5, Lemma 2]) from sets to
functions. The proof remains the same (as that of “regularity plus Lindelöfness
implies normality”).
**Lemma 2.1 (NORMALIZATION LEMMA).** Let \( X \) be a topological space and \( a \leq b \) in \( F(X) \). Let \( \mathcal{G} \subseteq LSC(X) \) and \( \mathcal{H} \subseteq USC(X) \) be countable families such that \( a \leq \bigvee \mathcal{G} \leq \bigvee \mathcal{G}^* \leq b \) and \( a \leq \bigwedge \mathcal{H} \leq \bigwedge \mathcal{H}^* \leq b \). Then there exists an \( f \in LSC(X) \) such that \( a \leq f \leq f^* \leq b \).

**Proof:** Let \( \mathcal{G} = \{ g_n : n \in \mathbb{N} \} \) and \( \mathcal{H} = \{ h_n : n \in \mathbb{N} \} \). Define \( f_1 = g_1 \) and \( f_n = g_n \land \bigwedge \{(h_i)_* : i < n\} \) for all \( n \geq 2 \). Then \( f = \bigvee \{ f_n : n \in \mathbb{N} \} \in LSC(X) \) has the required properties. Indeed, we have

\[
a \leq \bigvee \mathcal{G} \land \bigwedge \mathcal{H} \leq g_1 \lor \bigvee_{n \geq 2} (g_n \land \bigwedge_{i < n} (h_i)_*) = f.
\]

Since \( f_m \leq \bigvee \{ g_i^* : i \leq n \} \) if \( m \leq n \), and \( f_m \leq h_n \) if \( m > n \), hence \( f_m \leq h_n \lor \bigvee \{ g_i^* : i \leq n \} \in USC(X) \) for all \( m \) and \( n \). Therefore

\[
f^* \leq \bigwedge_{n \in \mathbb{N}} (h_n \lor \bigvee g_i^*) \leq \bigwedge \mathcal{H} \lor \bigvee \mathcal{G}^* \leq b.
\]

\( \square \)

**Remark 2.2.** Let us note the following obvious facts. Given \( a, b \in F(X) \), \( \mathcal{A} \subseteq F(X) \), and \( t \in \mathbb{R} \) one has:

1. \( \bigvee \mathcal{A} > t = \bigcup \{ [a \geq t + 1/n] : a \in \mathcal{A} \text{ and } n \in \mathbb{N} \} \).
2. \( \bigwedge \mathcal{A} < t = \bigcup \{ [a \leq t - 1/n] : a \in \mathcal{A} \text{ and } n \in \mathbb{N} \} \).
3. If \( a^* \leq b \) and \( a \leq b_* \), then \( [b < t] \subset [a^* < t] \subset [a \leq t] \) and \( [b < t] \subset [b_* \leq t] \subset [a \leq t] \), i.e. \( [a > t] \) and \( [b < t] \) are separated sets.

It follows from 2.1 that each two separated \( F_\sigma \)-sets in a normal space have disjoint open neighbourhoods (cf. [1, Chap. I, § 5, Lemma 3]), a fact which we need in proving 2.3 which follows. Note that 2.3 (2) can be obtained by combining 2.1 and the Katětov-Tong theorem. However, rather than refer to this theorem, we give an independent argument, and then deduce the Katětov-Tong theorem from the condition (2) of 2.3.

Let us denote by \( USC_\sigma(X) \) and \( LSC_\delta(X) \), respectively, the collections of ‘sup’s and ‘inf’s of all the countable families of \( USC(X) \) and \( LSC(X) \). (Note that if \( a \in USC_\sigma(X) \) and \( b \in LSC_\delta(X) \), then \( a \leq b \) implies that both \( a \) and \( b \) are in \( F(X) \). In actual fact, all those warnings we make about the finiteness of the involved functions do not really matter, because all the results of this paper are valid for extended functions.)

**Theorem 2.3.** For \( X \) a topological space, the following statements are equivalent:

1. \( X \) is normal.
2. If \( a \in USC_\sigma(X) \), \( b \in LSC_\delta(X) \), \( a^* \leq b \) and \( a \leq b_* \), then there exists an \( f \in LSC(X) \) such that \( a \leq f \leq f^* \leq b \).
3. (KATĚTOV-TONG THEOREM) If \( a \in USC(X) \), \( b \in LSC(X) \), and \( a \leq b \), then there exists an \( f \in C(X) \) such that \( a \leq f \leq b \).
Proof: (1) \(\Rightarrow\) (2): By 2.2, \([a > r]\) and \([b < r]\) are separated \(F_\sigma\)-sets for each \(r \in \mathbb{Q}\). By normality, for each \(r \in \mathbb{Q}\) there exists an open set \(U_r\) such that

\[\tag{*} [a > r] \subset U_r \subset \overline{U}_r \subset [b \geq r].\]

We first consider the case in which \(a\) and \(b\) are \(I\)-valued. By \((*)\) with \(\mathcal{G} = \{r \land \chi_{U_r} : r \in \mathbb{Q} \cap I\} \subset \text{LSC}(X)\) and \(\mathcal{H} = \{r \lor \chi_{\overline{U}_r} : r \in \mathbb{Q} \cap I\} \subset \text{USC}(X)\) one has

\[a = \bigvee_{r \in \mathbb{Q} \cap I} r \land \chi_{[a > r]} \leq \bigvee_{r \in \mathbb{Q} \cap I} \mathcal{G} \leq \bigvee_{r \in \mathbb{Q} \cap I} \mathcal{G}^* \leq \bigvee_{r \in \mathbb{Q} \cap I} r \land \chi_{[b \geq r]} = b\]

and

\[a = \bigwedge_{r \in \mathbb{Q} \cap I} r \lor \chi_{[a > r]} \leq \bigwedge_{r \in \mathbb{Q} \cap I} \mathcal{H} \leq \bigwedge_{r \in \mathbb{Q} \cap I} \mathcal{H}^* \leq \bigwedge_{r \in \mathbb{Q} \cap I} r \lor \chi_{[b \geq r]} = b.\]

By 2.1, there exists an \(f\) in \(\text{LSC}(X)\) with \(a \leq f \leq f^* \leq b\).

Suppose now that \(a\) and \(b\) are \(\mathbb{R}\)-valued. Let \(h : \mathbb{R} \to (0, 1)\) be an increasing homomorphism. Since \(h\) is inf- and sup-preserving, we have \((h \circ a)^* = h \circ a^*\) and \((h \circ b)_* = h \circ b_*\) with \(h \circ a \in \text{USC}_\sigma(X)\) and \(h \circ b \in \text{LSC}_\delta(X)\) being \((0, 1)\)-valued.

By what was just proved, there is an \(f\) in \(\text{LSC}(X)\) such that \(h \circ a \leq f \leq f^* \leq h \circ b\) and, thus, with values in \((0, 1)\). Then \(h^{-1} \circ f\) is the required function.

(2) \(\Rightarrow\) (3): As above, we may assume that \(a\) and \(b\) take values in \(I\). By (2) there exists \(g_0 \in \text{LSC}(X)\) such that \(a \leq g_0 \leq g_0^* \leq b = g_1\). A procedure perfectly analogous to that in the Urysohn’s lemma construction (e.g. [6, 1.5.10]) can now be used to define a family \(\{g_r : r \in \mathbb{Q} \cap I\} \subset \text{LSC}(X)\) such that

\[g_0 \leq g_r \leq g_1\text{ and } g_r^* \leq g_s\text{ whenever } r < s.\]

Next, define

\[U_r = [g_r > 1 - r]\text{ if } 0 \leq r \leq 1,\]
\[= \emptyset\text{ if } r < 0,\]
\[= X\text{ if } r > 1.\]

Then each \(U_r\) is open and \(\overline{U}_r \subset U_s\) whenever \(r < s\) in \(\mathbb{Q}\). For if \(r < s\) in \(\mathbb{Q} \cap I\), then \(U_r \subset [g_r^* \geq 1 - r] \subset U_s\). Further, it is well-known that \(g\) defined by \(g(x) = \inf\{r \in \mathbb{Q} : x \in U_r\}\), \(x \in X\), is continuous, and clearly takes values in \(I\). Since \(a \leq g_r \leq b\), hence \([a > r - 1] \subset U_r \subset [b > r - 1]\) for all \(r\), and we conclude that \(1 - b \leq g \leq 1 - a\). Therefore \(f = 1 - g\) is the required function.

(3) \(\Rightarrow\) (1): Obvious.  \(\Box\)

We note that the proof of the implication (1) \(\Rightarrow\) (2) of 2.3 merely depends upon the condition that \([a > r]\) and \([b < r]\) have disjoint open neighbourhoods. Since this condition is trivially a necessary one for the insertion of \(f\) and \(f^*\) between \(a\) and \(b\), we thus have the following
Theorem 2.4. Let $X$ be a topological space. For $a \leq b$ in $F(X)$, the following are equivalent:

1. There exists an $f \in LSC(X)$ such that $a \leq f \leq f^* \leq b$.
2. For every $r$ in $\mathbb{Q}$, $[a > r]$ and $[b < r]$ have disjoint open neighbourhoods.

Note in passing that the condition in the normalization lemma is another one which is necessary and sufficient.

One may invent further special cases of 2.4 apart from 2.3(2). The following one is of some interest, since it characterizes complete normality in terms of real functions and seems to be new in the metric context as well. A self-contained proof based on a more traditional method is given in [11, Theorem 1].

Corollary 2.5 ([11]). Let $X$ be a topological space. The following conditions are equivalent:

1. $X$ is completely normal.
2. If $a, b \in F(X)$, $a^* \leq b$ and $a \leq b^*$, then there exists an $f \in LSC(X)$ such that $a \leq f \leq f^* \leq b$.

Proof: (1) $\Rightarrow$ (2): By 2.2(3), $[a > r]$ and $[b < r]$ are separated for each $r \in \mathbb{Q}$, hence have disjoint open neighbourhoods by complete normality.

(2) $\Rightarrow$ (1): As in [11], if $A, B \subset X$ are such that $\overline{A} \subset B$ and $A \subset \text{Int}B$, then $(\chi_A)^* \leq \chi_B$ and $\chi_A \leq (\chi_B)^*$. Given $f \in LSC(X)$ with $\chi_A \leq f \leq f^* \leq \chi_B$, one gets $A \subset U \subset \overline{U} \subset B$ with open $U = [f > \frac{1}{2}]$.

Remark 2.6. Both 2.3 and 2.5 may have a fuzzy topological interpretation. With $X$ a topological space, $(X, LSC(X, I))$ is a fuzzy topological space, and 2.3 states that $X$ is normal iff $(X, LSC(X, I))$ is fuzzy normal, while 2.5 states that $X$ is completely normal iff $(X, LSC(X, I))$ is fuzzy completely normal. Whether it is so in the latter case, was an open question by the present author in [16].

References


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