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A note on the existence of solution for semilinear heat equations with polynomial growth nonlinearity*

WAN SE KIM

Abstract. The existence of weak solution for periodic-Dirichlet problem to semilinear heat equations with superlinear growth non-linear term is treated.

Keywords: periodic-Dirichlet problem, semilinear heat equation, superlinear growth

Classification: 35K05, 35K20

1. Introduction.

Let Z^+ , Z and R be the set of all positive integers, integers and real numbers, respectively, and let $\Omega = [0, 2\pi] \times [0, \pi]$ and $I = [0, \pi]$. Let $p \in [1, \infty)$; by $L^p(\Omega)$ we denote the space of all measurable real valued functions $u(t, x)$ for which $|u(t, x)|^p$ is Lebesgue integrable.

The norm is given by

$$\|U\|_{L^p} = \left[\int \int_{\Omega} |u(t, x)|^p dt dx \right]^{1/p}.$$

In particular, $L^2(\Omega)$ is a Hilbert space having the usual inner product $\langle \cdot, \cdot \rangle$ and the usual norm $\|\cdot\|_{L^2}$. Let $C^k(\Omega)$ be the space of all continuous functions $u(t, x)$ such that the partial derivatives up to order k with respect to both variables exist and are continuous on Ω , while $C(\Omega)$ is used for $C^0(\Omega)$ with the usual norm $\|\cdot\|$ and we write $C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$.

Consider the periodic-Dirichlet problem

$$(1.1) \quad u_t(t, x) - u_{xx}(t, x) + g(x, u(t, x)) = h(t, x) \quad \text{in } \Omega$$

$$(1.2) \quad \begin{aligned} u(0, x) - u(2\pi, x) &= u_t(0, x) - u_t(2\pi, x) = 0 \quad \text{for all } x \in I \\ u|_{\partial I} &= 0, \end{aligned}$$

where $g : I \times R \rightarrow R$ is a Carathéodory function, that is, $g(\cdot, u)$ is measurable on I for each $u \in R$ and $g(x, \cdot)$ is continuous on R with the continuity uniform with respect to a.e. $x \in I$. This holds, for example, if g is continuous on $I \times R$, but

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it holds in many other cases; e.g. $g(x, u) = p(x)F(u)$, where $p \in L^\infty(I)$ and F is continuous. Moreover, we assume that for each $d > 0$ there exists a constant $M_d > 0$ such that $|g(x, u)| \leq M_d$ for all $(x, u) \in I \times R$ with $|u| \leq d$ and $h \in L^2(\Omega)$. Let $H^1(\Omega)$ be the completion of the space $C^\infty(\Omega)$ with respect to the norm given by

$$(1.3) \quad \|u\|_1^2 = \int \int_\Omega [|u^2(t, x)|^2 + |u_t(t, x)|^2 + |u_x(t, x)|^2] dt dx$$

and $H^{1,2}(\Omega)$ be the completion of the space $C^\infty(\Omega)$ with respect to the norm given by

$$(1.4) \quad \|u\|_{1,2}^2 = \int \int_\Omega [|u(t, x)|^2 + |u_t(t, x)|^2 + |u_x(t, x)|^2 + |u_{xx}(t, x)|^2] dt dx.$$

Note that $H^1(\Omega)$ ($H^{1,2}(\Omega)$) has distributional derivatives $u_t, u_x \in L^2(\Omega)$ ($u_t, u_x, u_{xx} \in L^2(\Omega)$) and these derivatives can be obtained as limit in $L^2(\Omega)$ of the corresponding derivatives of a sequence of $C^\infty(\Omega)$ functions which tend to u in $H^1(\Omega)$ ($H^{1,2}(\Omega)$). Moreover if derivatives are interpreted in distributional sense, the norms in $H^1(\Omega)$ and $H^{1,2}(\Omega)$ are given by (1.3) and (1.4), respectively, and $H^{1,2}(\Omega) \subseteq C(\Omega)$, and the imbedding of $H^{1,2}(\Omega)$ in $C(\Omega)$ is continuous. Let $H^{0,1}(\Omega)$ be the closure in $H^1(\Omega)$ of all functions $u(t, x)$ in $C^\infty(\Omega)$.

A weak solution to the periodic-Dirichlet problem (1.1), (1.2) on Ω will be $u \in H^{1,2}(\Omega) \cap H^{0,1}(\Omega)$ which satisfies the equation (1.1) a.e. on Ω and the boundary condition (1.2).

Several authors deal with the periodic-Dirichlet problem for semilinear heat equations. For example, Brezis and Nirenberg [1], Fučík [2], Nkashama and Willem [3], Šťastnová and Fučík [5] and the references in [6]. In particular, Sanchez [4] shows the existence of solutions for some power-like behavior of g by using Leray-Schauder principle.

In this note, we will investigate the existence of weak solution of the problem (1.1) (1.2) when g has polynomial growth and satisfies a sign condition. The rate of growth allowed in g is any polynomial growth; i.e. there exist $a(x), b(x)$ in $L^\infty(\Omega)$ such that

$$(H_1) \quad |g(x, u)| \leq a(x)|u|^p + b(x) \text{ for } x \in I, |u| \geq d_0 \text{ and } p > 0.$$

The sign condition on g used here is that there exists a function $\psi : R \rightarrow R$ such that $\limsup_{|u| \rightarrow \infty} \psi(u)/u = \alpha_0$ exists with $ug(x, u) \geq -\psi(|u|)$ for all $(x, u) \in I \times R$. We do not need any restriction on h except $h \in L^2(\Omega)$ and our proof is based on the use of Fourier series and Leray-Schauder's continuation theorem by finding an a priori bound for all possible solutions of the associated equations.

2. Preliminaries.

Let $Dom L = \{u \in H^{1,2}(\Omega) \cap H^{0,1}(\Omega) \mid u \text{ satisfies (1.2)}\}$ and define the linear operator

$$(2.1) \quad \begin{aligned} L : Dom L \subseteq C(\Omega) &\rightarrow L^2(\Omega) \text{ by} \\ Lu &= u_t - u_{xx}. \end{aligned}$$

Then $Ker L = \{0\}$ and $Im L = L^2(\Omega)$. Hence $L^{-1} : L^2(\Omega) \rightarrow Dom L$ exists. Since we may expand function u and h satisfying (1.2) on Ω in the following form:

$$u(t, x) = \sum_{(l,m) \in Z \times Z^+} u_{lm} \exp(ilt) \sin(mx)$$

$$h(t, x) = \sum_{(l,m) \in Z \times Z^+} h_{lm} \exp(ilt) \sin(mx)$$

with $u_{lm} = u_{-l-m}$ and $h_{lm} = h_{-l-m}$ since u and h are real, the abstract actions of L and L^{-1} are determined as follows.

$$(2.2) \quad Lu(t, x) = \sum_{(l,m) \in Z \times Z^+} [li + m^2] u_{lm} \exp(ilt) \sin(mx)$$

and

$$(2.3) \quad L^{-1}u(t, x) = \sum_{(l,m) \in Z \times Z^+} [li + m^2]^{-1} h_{lm} \exp(ilt) \sin(mx).$$

Since

$$h_{lm} = \frac{1}{2\pi^2} \int \int_{\Omega} h(s, y) \exp(-ilt) \sin(mx) dt dx,$$

we can represent L^{-1} as a convolution product.

$$(2.4) \quad L^{-1}h(t, x) = \int \int_{\Omega} k(t, s, x, y) h(s, y) ds dy,$$

where

$$k(t, s, xy) = \frac{1}{2\pi^2} \sum_{(l,m) \in Z \times Z^+} [li + m^2] \exp[il(t - s)] \sin(mx) \sin(my).$$

Lemma. The operator $L^{-1} : L^2(\Omega) \rightarrow C(\Omega)$ is a compact operator and $\|L^{-1}\|_{\infty} \leq C \|h\|_{L^2}$ for some constant $C > 0$ independently of h .

PROOF: Using (2.4) and the convergence $\sum_{(l,m) \in Z \times Z^+} [l^2 + m^4]^{-1}$ and Arzela-Ascoli theorem, we can prove our assertion.

□

3. Main results.

Theorem. *Let $h \in L^2(\Omega)$ and suppose that (H_1) is satisfied and*

(H_2) for all $(x, u) \in I \times R, ug(x, u) \geq -\psi(|u|)$, where $\psi : R \rightarrow R$ is a function such that

$$\limsup_{|u| \rightarrow \infty} \psi(u)/u = \alpha_0, \quad \alpha_0 \in R,$$

exists, then the periodic-Dirichlet problem on Ω for the equation (1.1) has at least one weak solution.

PROOF: Now we see that $L^{-1} : L^2(\Omega) \rightarrow C(\Omega)$ is continuous and compact operator as a mapping from $L^2(\Omega)$ into $C(\Omega)$. Define a substitution operator $N : C(\Omega) \rightarrow L^2(\Omega)$ by $Nu(t, x) = -g(x, u(t, x)) + h(t, x)$ for all $u \in C(\Omega)$ and $(t, x) \in \Omega$. These definitions have a meaning because $H^{1,2}(\Omega) \cap H^1(\Omega)$ are continuously imbedded in $C(\Omega)$. Then u is weak solution of the periodic-Dirichlet problem for (1.1) if and only if $u \in Dom L$ and satisfies

$$(3.1) \quad Lu = Nu, \quad \text{or equivalently}$$

$$(3.2) \quad u = L^{-1}Nu.$$

If $u \in C(\Omega)$ solves the operator equation (3.2), then $u \in C(\Omega)$ is a weak solution to the periodic-Dirichlet problem. Since L^{-1} is compact, and N is continuous and maps bounded sets into set bounded sets, the composition $L^{-1}N : C(\Omega) \rightarrow C(\Omega)$ is continuous and compact. By using Leray-Schauder theory if all solutions u to the family of equations

$$(3.3) \quad u = \lambda L^{-1}Nu, \quad 0 \leq \lambda \leq 1$$

are bounded in $C(\Omega)$ independently of $\lambda \in [0, 1]$, then (3.1) has a solution.

If (u, λ) solves (3.3), then (u, λ) solves

$$(3.4) \quad Lu = \lambda Nu$$

and u is a weak solution to the periodic-Dirichlet problem of the equation

$$u_t - u_{xx} + \lambda g(x, u) = \lambda h(t, x) \quad \text{on } \Omega.$$

Thus the proof will be completed if we show that the solutions to (3.4) for $0 \leq \lambda \leq 1$ are bounded in $C(\Omega)$ independently of $\lambda \in [0, 1]$. Since, if $\lambda = 0$, we have only the trivial solution $u \equiv 0$, it suffices to show our assertion for $0 < \lambda \leq 1$.

To this end, let (u, λ) be any solution to (3.4) with $0 < \lambda \leq 1$. By taking the inner product with u_t on both sides of (3.4), we obtain

$$\langle Lu, u_t \rangle + \lambda \langle g(x, u), u_t \rangle = \lambda \langle h, u_t \rangle.$$

Since $u \in Dom L$, there exists a sequence $u_n \in C^\infty(\Omega)$, u_n satisfy (1.2), such that the distributional derivatives u_t, u_x, u_{xx} can be obtained as limits in $L^2(\Omega)$ of

the corresponding derivatives of u_n which tend to u in $H^{1,2}(\Omega)$. Now integration of these smooth functions, using the boundary conditions, shows that for each $n \in Z^+$, $\langle Lu_n, u_{nt} \rangle = \|u_{nt}\|_{L^2}$. Letting $n \rightarrow +\infty$, we obtain $\langle Lu, u_t \rangle = \|u_{nt}\|_{L^2}$.

Moreover, since for each $n \in Z^+$, the periodicity of $u_n(t, x)$ in t implies $\langle g(x, u_n), u_{nt} \rangle = 0$, we also have $\langle g(x, u), u_t \rangle = 0$. Thus

$$(3.5) \quad \|u_t\|_{L^2} \leq \|h\|_{L^2}.$$

Next we prove that $\|u\|_{L^2} \leq M_1$ for some $M_1 > 0$ independently of $[0, 1]$. Since $\limsup_{|u| \rightarrow \infty} \psi(u)/u = \alpha_0$, for $\alpha \geq 0$ with $\alpha > \alpha_0$, there exists $r_0 > 0$ such that $\psi(u)/u \leq \alpha$ with $|u| > r_0$. So $\psi(u) \leq \alpha u$ for all $|u| \leq r_0$. Thus

$$\int \int_{|u| > r_0} ug(x, u) dt dx \geq -2^{1/2} \alpha \pi \|u\|_{L^2}.$$

Since g is a Carathéodory function on $I \times R$, there exists a constant $M_{r_0} > 0$ such that $|g(x, u)| \leq M_{r_0}$ for u with $|u| \leq r_0$ for $x \in I$. Hence

$$\left| \int \int_{|u| \leq r_0} ug(x, u) dt dx \right| \leq 2\pi^2 r_0 M_{r_0}.$$

Thus

$$\begin{aligned} \langle g(x, u), u \rangle &= \int \int_{|u| > r_0} ug(x, u) dt dx + \int \int_{|u| \leq r_0} ug(x, u) dt dx \\ &\geq -2^{1/2} \alpha \pi \|u\|_{L^2} - 2\pi^2 r_0 M_{r_0}. \end{aligned}$$

By taking the inner product with u on both sides of (3.4), we have, by the similar argument

$$\|u_{nx}\|_{L^2}^2 + \lambda \langle g(x, u), u \rangle = \lambda \langle h, u \rangle$$

since

$$\langle u_t, u \rangle = 0 \quad \text{and} \quad \langle u_{xx}, u \rangle = \|u_{nx}\|_{L^2}^2.$$

Thus

$$(3.6) \quad \begin{aligned} \|u_x\|_{L^2}^2 &\leq -\langle g(x, u), u \rangle + \langle h, u \rangle \\ &\leq [2^{-1/2} \alpha \pi + \|h\|_{L^2}] \|u\|_{L^2} + 2\pi^2 r_0 M_{r_0}. \end{aligned}$$

But since $\|u\|_{L^2} \leq c_1 \|u_x\|_{L^2}$ for all $u \in Dom L$ and for some $c_1 > 0$,

$$\|u\|_{L^2}^2 \leq C_1^2 [2^{-1/2} \alpha \pi + \|h\|_{L^2}] \|u\|_{L^2} + 2\pi^2 r_0 M_{r_0}.$$

Therefore, there exists a constant M_1 independently of $\lambda \in (0, 1]$ such that

$$(3.7) \quad \|u\|_{L^2} \leq M_1.$$

Hence, by (3.6), there exists a constant M_2 independently of $\lambda \in (0, 1]$ such that

$$(3.8) \quad \|u\|_{L^2} \leq M_2.$$

Hence, we have $u \in H^1(\Omega)$ and $\|u\|_{H^1} \leq L_1$ for some constant independently of $\lambda \in (0, 1]$. Since $L^2(\Omega) \subseteq L^q(\Omega)$ where $1 \leq q \leq 2$ and since $H^1(\Omega)$ is embedded in $L^q(\Omega)$ where $2 \leq q \leq \infty$ (see e.g. [2], [6]), $\|u\|_{L^q} \leq L_2(q)$ where $L_2(q)$ may depend on L_1 and $q \geq 1$ but is independent of $\lambda \in (0, 1]$.

Next, we will estimate the L^2 -bound for $g(\cdot, u)$. For $|u| \leq d_0 + 1, x \in I$, we have

$$|g(x, u)| \leq \sup_{\substack{x \in I \\ |u| \leq d_0 + 1}} |g(x, u)| \leq M_3$$

for some $M_3 > 0$ since g is a Carathéodory function. For $|u| > d_0 + 1, x \in I$, we have, by (H_1) ,

$$|g(x, u)| = (1/|u|)|ug(x, u)| \leq (1/d_0)[a(x)|u|^{p+1} + b(x)|u|].$$

Therefore, we have

$$\begin{aligned} |g(x, u)| &\leq \sup_{\substack{x \in I \\ |u| \leq d_0 + 1}} |g(x, u)| + \sup_{\substack{x \in I \\ |u| > d_0 + 1}} (1/|u|)|ug(x, u)| \\ &\leq (1/d_0)[a(x)|u|^{p+1} + b(x)|u|] + M_3. \end{aligned}$$

Hence

$$\begin{aligned} \|g(\cdot, u)\|_{L^2}^2 &\leq M_4 \|u\|_{L^{2p+2}}^{2p+2} + M_5 \|u\|_{L^{p+2}}^{p+2} \\ &\quad + M_6 \|u\|_{L^{p+1}}^{p+1} + M_7 \|u\|_{L^2}^2 + M_8 \|u\|_{L^2} + M_9 \end{aligned}$$

for some appropriate constants M_4, M_9 .

Since $\|u\|_{L^q} \leq L_2(q)$ for $q \geq 1, \|g(\cdot, u)\|_{L^2} \leq L_0$ for some L_0 independently of $\lambda \in [0, 1]$.

So if (u, λ) is any solution to (3.3), then, by Lemma,

$$\|u\|_\infty \leq \|L^{-1}Nu\|_\infty \leq C\|g(\cdot, u)\|_{L^2} + \|h\|_{L^2} \leq C(L_0 + \|h\|_{L^2})$$

and this completes our proof. □

Corollary. For any $h \in L^2(\Omega)$ and $F \in L^\infty(I)$ the boundary value problem

$$\begin{aligned} u_t - u_{xx} + F(x) \operatorname{sgn}(u)|u|^p &= h(t, x), \quad p > 0 \\ u(0, x) - u(2\pi, x) &= u_t(0, x) - u_t(2\pi, x) = 0 \quad \text{for all } x \in I \\ u \mid \partial I &= 0 \end{aligned}$$

has at least one weak solution.

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