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Totally bounded frame quasi-uniformities

P. Fletcher, W. Hunsaker, W. Lindgren

Abstract. This paper considers totally bounded quasi-uniformities and quasi-proximities for frames and shows that for a given quasi-proximity \( \bowtie \) on a frame \( L \) there is a totally bounded quasi-uniformity on \( L \) that is the coarsest quasi-uniformity, and the only totally bounded quasi-uniformity, that determines \( \bowtie \). The constructions due to B. Banaschewski and A. Pultr of the Cauchy spectrum \( \psi L \) and the compactification \( \mathcal{R}L \) of a uniform frame \( (L, U) \) are meaningful for quasi-uniform frames. If \( U \) is a totally bounded quasi-uniformity on a frame \( L \), there is a totally bounded quasi-uniformity \( \mathcal{U} \) on \( \mathcal{R}L \) such that \( (\mathcal{R}L, \mathcal{U}) \) is a compactification of \( (L, U) \). Moreover, the Cauchy spectrum of the uniform frame \( (Fr(U^*), U^*) \) can be viewed as the spectrum of the bicompletion of \( (L, U) \).

Keywords: frame, uniform frame, quasi-uniform frame, quasi-proximity, totally bounded quasi-uniformity, uniformly regular ideal, compactification, bicompletion

Classification: 6D20, 18B35, 54D35, 54E05, 54E15

0. Introduction.

The concept of a quasi-proximity for a topological space was introduced by C.H. Dowker [4]. In [12] W. Hunsaker and W. Lindgren proved that there is a one-to-one correspondence between quasi-proximities and totally bounded quasi-uniformities and that each quasi-proximity class of quasi-uniformities contains a coarsest member, which is totally bounded. In this paper, we introduce the concept of a frame quasi-proximity, obtain results for frames analogous to those obtained for spaces in [12], and discuss compactifications of totally bounded quasi-uniform frames.

Let \( U \) be a totally bounded quasi-uniformity and let \( L \) be the frame determined by \( U^* \). In [3] B. Banaschewski and A. Pultr give a compactification \( \mathcal{R}L \) of the uniform frame \( (L, U^*) \). We show that there exists a totally bounded quasi-uniformity \( \mathcal{U} \) on \( \mathcal{R}L \) such that \( \mathcal{U}^* \) determines \( \mathcal{R}L \) and that there exists a dense quasi-uniform frame homomorphism from \( (\mathcal{R}L, \mathcal{U}) \) onto \( (L, U) \).

In the last section we consider briefly another construction from [3], the Cauchy spectrum of a uniform frame. We show that if \( U \) is a quasi-uniformity then the Cauchy spectrum of the underlying uniform frame \( (Fr(U^*), U^*) \) can be constructed directly from the quasi-uniformity \( U \) in a manner that parallels the construction of the bicompletion of a quasi-uniform space [9].

1. Preliminaries.

A frame \((L, \leq)\) is a complete lattice that satisfies the frame distributive law: \( a \wedge \bigvee S = \bigvee \{ a \wedge x \mid x \in S \} \) for any \( a \in L \) and any \( S \subseteq L \). A function \( f : L \to M \)
between frames is a **join homomorphism** provided that for any \( S \subseteq L \), \( f(\bigvee S) = \bigvee \{ f(s) : s \in S \} \). A join homomorphism that also preserves finite meets is called a **frame homomorphism**. We use 1 to denote \( \bigwedge \emptyset \) and 0 to denote \( \bigvee \emptyset \). A subset \( C \) of a frame \((L, \leq)\) is a **cover** provided that \( \bigvee C = 1 \). For each \( a \in L \), \( \overline{a} \) denotes \( \bigvee \{ x \in L : x \wedge a = 0 \} \); this element \( \overline{a} \) is called the **pseudocomplement** of \( a \). Throughout this paper if \( F \) is a collection of functions mapping a frame \( L \) to a frame \( M \) we define \( \bigwedge F \) pointwise and for \( u, v \in F \) we write \( u \leq v \) to mean that for each \( x \in L \), \( u(x) \leq v(x) \).

We recall the following fundamental concepts and results from [8].

For \( a \) and \( b \) in \( L \), the function \( a \# b : L \to L \) is defined by

\[
a \# b(x) = \begin{cases} 
  b & \text{if } a \wedge x \neq 0 \\
  0 & \text{otherwise}.
\end{cases}
\]

If \( u : L \to L \) is any function and \( x \in L \), then \( x \) is **\( \# \)-small** provided that \( x \# x \leq u \). The collection of all \( \# \)-small elements is denoted by \( S_u \), and if \( u \) is an order-preserving function such that \( \bigvee S_u = 1 \) we say that \( u \) is a **\( \Delta \)-map**.

A **frame quasi-uniformity base** supported on a frame \((L, \leq)\) is a collection \( B \) of \( \Delta \)-maps such that

(1) For each \( u \in B \) there exists \( v \in B \) such that \( v \circ v \leq u \).

(2) For \( u, v \in B \) there is a join homomorphism \( w \) and a \( z \in B \) such that \( z \leq w \leq u \wedge v \).

If \( B \) is a frame quasi-uniformity base, then the quasi-uniformity \( U \) for which \( B \) is a base is the collection of all \( w : L \to L \) such that \( w \) is order preserving and there is a \( u \in B \) with \( u \leq w \). The members of a quasi-uniformity \( U \) are called **entourages**. If \( B \) satisfies:

(3) For each \( u \in B \) and for each \( x, y \in L, u(x) \wedge y = 0 \) if and only if \( u(y) \wedge x = 0 \),

then \( B \) is a **base for a frame uniformity for \( L \)**.

A collection \( D \) of \( \Delta \)-maps is a **subbase** for a frame quasi-uniformity \( U \) provided that the collection of all finite meets from \( D \) is a base for \( U \).

The frame of \( U \), denoted by \( Fr(U) \) is the collection to which \( a \) belongs provided that

\[ a = \bigvee \{ b \in L : u(b) \leq a \text{ for some } u \in U \}. \]

We say that \( U \) **determines \( L \)** provided that \( Fr(U) = L \).

Let \( U \) and \( V \) be quasi-uniformities on frames \( L \) and \( M \) respectively and let \( f : L \to M \) be a frame homomorphism. Then \( f \) is a **quasi-uniform frame homomorphism** provided that for every \( u \in U \) there exists a \( v \in V \) such that \( v \circ f \leq f \circ u \).

For each \( \Delta \)-map \( u \) and each \( x \in L \) define

\[
\widehat{u} : L \to L \text{ by } \widehat{u}(x) = \bigvee \{ b \# a : a \# b \leq u \}(x)
\]

and

\[
u^* : L \to L \text{ by } u^*(x) = \bigvee \{ a : a \# a \leq u \text{ and } a \wedge x \neq 0 \}.
\]
Then for any quasi-uniformity $U$ supported on a frame $L$, $\{\hat{u} : u \in U\}$ is a base for a quasi-uniformity $\hat{U}$ on $L$ and $\{u^* : u \in U\}$ is a base for a uniformity $U^*$ on $L$ that is the coarsest quasi-uniformity containing $U \cup \hat{U}$. The underlying biframe of $U$ is the triple $(Fr(U^*), Fr(U), Fr(\hat{U}))$. It is shown in [8] that the underlying biframe of $U$ is a biframe in the sense of B. Banaschewski, G.C.L. Brümmer and K. Hardie [2]. If $U$ is a quasi-uniformity on $L$ and $U^*$ determines $L$, we say that $(L, U)$ is a quasi-uniform frame.

2. Quasi-proximities.

In this section we extend the theory of quasi-proximities established in [12] to a theory of quasi-proximities for frames.

**Definition.** Let $(L, \leq)$ be a frame. A quasi-proximity on $L$ is a binary relation $\alpha$ on $L$ satisfying the following axioms for $a, b, c, d$ in $L$.

1. $0 \leq 0$ and $1 \leq 1$.
2. If $a \leq b$, then $a \leq b$.
3. If $a \leq b \leq c \leq d$, then $a \leq d$.
4. If $a \leq b$ and $a \leq c$, then $a \leq b \land c$.
5. If $a \leq c$ and $b \leq c$, then $a \lor b \leq c$.
6. If $a \leq b$, then there exists $c \in L$ such that $a \leq c \land b$.
7. If $a \leq b$, then $\overline{a} \lor b = 1$.

**Proposition 2.1.** Let $(L, \leq)$ be a frame and let $U$ be a quasi-uniform base on $L$. For $a, b \in L$ define $a \triangleleft b$ if and only if $u(a) \leq b$ for some $u \in U$. Then $\triangleleft$ is a quasi-proximity on $L$.

**Proof:** The axioms (1) – (5) follow easily from the properties of a quasi-uniformity and axiom (6) holds as in the proof of [8, Proposition 5.1]. To see that axiom (7) holds suppose that $a \triangleleft b$ and let $u \in U$ such that $u(a) \leq b$. It suffices to show that $\overline{a} \lor u(a) = 1$. We have $1 = \bigvee \{x \in L : x \text{ is } u\text{-small}\} = \bigvee \{x \in L : x \text{ is } u\text{-small and } x \land a \neq 0\} \lor \bigvee \{x \in L : x \text{ is } u\text{-small and } x \land a = 0\} \leq u(a) \lor \overline{a}$. □

**Definition.** If $U$ is a quasi-uniformity (base) on a frame $L$, then the quasi-proximity $\triangleleft$ defined by $a \triangleleft b$ if and only if $u(a) \leq b$ for some $u \in U$ is called the quasi-proximity determined by $U$.

**Lemma 2.2.** Let $(L, \leq)$ be a frame. Let $C = \{(a_\alpha, b_\alpha) : a_\alpha, b_\alpha \in L, \alpha \in A\}$ and suppose that for each $B \subseteq A$, $(\bigwedge_{\alpha \in B} a_\alpha, \bigwedge_{\alpha \in B} b_\alpha) \in C$ and $(\bigvee_{\alpha \in B} a_\alpha, \bigvee_{\alpha \in B} b_\alpha) \in C$. For each $\alpha \in A$ and each $x \in L$, let

$$u_\alpha(x) = \begin{cases} 0 & \text{if } x = 0 \\ b_\alpha & \text{if } x \leq a_\alpha \text{ and } x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

and let $u(x) = \bigwedge u_\alpha(x)$. Then $u : L \to L$ is a join homomorphism.

**Proof:** Let $x = \bigvee x_i$. Then for each $\alpha \in A$ and each $i$, $u_\alpha(x) \geq u_\alpha(x_i)$ and so $u(x) \geq \bigvee_i u(x_i)$. In order to show that $u(x) \leq \bigvee_i u(x_i)$ we may suppose that for
each \( i, u(x_i) \neq 1 \) and for some \( i, u(x_i) \neq 0 \). For each \( i \), let \( B_i = \{ \alpha : x_i \leq a_\alpha \} \). Then \( B_i \neq \emptyset \).

Let \( w_i = \bigwedge \{ a_\alpha : \alpha \in B_i \} \), \( z_i = \bigwedge \{ b_\alpha : \alpha \in B_i \} \). Then for each \( i \), \((w_i, z_i) \in C\), \( x_i \leq w_i \) and \( u(x_i) = z_i \). Let \( w = \bigvee w_i \) and let \( z = \bigvee z_i \). Then \((w, z) \in C\); hence \((w, z) = (a_\gamma, b_\gamma)\) for some \( \gamma \in A \) and \( u(x) \leq u_\gamma(x) = z = \bigvee_i u(x_i) \).

\[ \square \]

**Definition.** Let \( L \) be a frame and let \( U \) be a quasi-uniformity on \( L \). Then \( U \) is **totally bounded** provided that for each \( u \in U \) there is a finite cover of \( L \) by \( u \)-small elements.

**Theorem 2.3.** Let \( L \) be a frame and let \( \vartriangleleft \) be a quasi-proximity on \( L \). For \( a, b \in L \) define

\[
\begin{align*}
    u_{a,b}(x) &= \begin{cases} 
        0 & \text{if } x = 0 \\
        b & \text{if } x \leq a, x \neq 0 \\
        1 & \text{otherwise}
    \end{cases} 
\end{align*}
\]

and let \( S = \{ u_{a,b} : a \triangleleft b \} \). Then \( S \) is a subbase for a totally bounded frame quasi-uniformity \( U_{\vartriangleleft} \), which determines \( \vartriangleleft \), and is the only totally bounded frame quasi-uniformity that determines \( \vartriangleleft \).

**Proof:** We first prove that \( S \) is a subbase for a quasi-uniformity. Let \( a, b \in L \) and suppose that \( a \triangleleft b \). Then \( a \) and \( b \) are \( u_{a,b} \)-small and so \( u_{a,b} \) is a \( \Delta \)-map. Let \( u_{a_i,b_i} \in S\), \( 1 \leq i \leq n \). Let \( D = \{(a_i, b_i) : 1 \leq i \leq n\} \) and form \( C = \{(a_\alpha, b_\alpha) : \alpha \in A\} \) by taking all meets and joins from \( D \). Let \( u = \bigwedge_{\alpha \in A} u_\alpha \) and note that \( u \leq \bigwedge_{i=1}^n u_{a_i,b_i} \).

It follows from Lemma 2.2 that \( u \) is a join homomorphism that is a finite meet of members of \( S \). Moreover, \( u \) is a \( \Delta \)-map.

Let \( u_{a,b} \in S \). There exists \( c \in L \) such that \( a \triangleleft c \triangleleft b \). Let \( w = u_{a,c} \wedge u_{c,b} \). It is easy to verify that \( w^2 \leq u_{a,b} \). Therefore \( S \) is a subbase for a frame quasi-uniformity \( U_{\vartriangleleft} \). If \( u_{a,b} \in S \), then \( \{a, b\} \) is a cover of \( L \) by \( u_{a,b} \)-small elements. It follows that \( U_{\vartriangleleft} \) is totally bounded.

We now show that \( U_{\vartriangleleft} \) determines \( \triangleleft \). Let \( \triangleleft_1 \) denote the quasi-proximity determined by \( U_{\vartriangleleft} \). Suppose that \( a \triangleleft_1 b \). Then \( u_{a,b}(a) \leq b \) and hence \( a \triangleleft_1 b \). Now suppose that \( a \triangleleft_1 b \). There exists \( u \in U_{\vartriangleleft} \) such that \( u(a) \leq b \). Since \( u \in U_{\vartriangleleft} \), there are \( (a_i, b_i) \), \( 1 \leq i \leq n \), such that \( a_i \triangleleft_1 b_i \) for each \( i \), and \( \bigwedge_{i=1}^n u_{a_i,b_i} \leq u \). Let \( w = \bigwedge_{i=1}^n u_{a_i,b_i} \). Let \( J = \{ i : a \leq a_i \} \), and let \( c = \bigwedge_{j \in J} a_j \), \( d = \bigwedge_{j \in J} b_j \). Then \( a \leq c \triangleleft d \leq b \).

We next show that \( U_{\vartriangleleft} \) is the coarsest frame quasi-uniformity that determines \( \triangleleft \). Suppose that \( V \) is a frame quasi-uniformity that determines \( \triangleleft \). Let \( u_{a,b} \in S \); then \( a \triangleleft b \) so there exists a join homomorphism \( v \in V \) such that \( v(a) \leq b \). It follows that \( v \leq u_{a,b} \).

Finally we show that \( U_{\vartriangleleft} \) is the only totally bounded frame quasi-uniformity that determines \( \triangleleft \). Suppose that \( V \) is a totally bounded frame quasi-uniformity that determines \( \triangleleft \). Let \( w \in V \) and let \( v \in V \) such that \( v^2 \leq w \). There exists a finite cover \( \{a_i\} \) of \( L \) by \( v \)-small elements. Since \( V \) determines \( \triangleleft \), we have that \( a_i \triangleleft v(a_i) \).
for all $i$. Note that $u_{a_i,v(a_i)} \in U_\triangleleft$ and let $z \in U_\triangleleft$ be a join homomorphism such that $z \leq \bigwedge_{i} u_{a_i,v(a_i)}$. To see that $z \leq w$ let $x \in L$. Then $z(x) = \bigvee_{i} z(x \wedge a_i)$. For each $j$,

$$z(x \wedge a_j) \leq \bigwedge_{i} u_{a_i,v(a_i)}(x \wedge a_j) \leq u_{a_j,v(a_j)}(a_j) \leq v(a_j) \leq v^2(x) \leq w(x).$$

\[ \square \]

### 3. Compactifications of totally bounded quasi-uniform frames.

Let $U$ be a totally bounded quasi-uniformity and let $(L, L_1, L_2)$ be the underlying biframe of $U$. Let $\triangleleft^*$ be the quasi-proximity determined by $U^*$. We note that $\triangleleft^*$ is the “uniformly below” relation of [3, p. 63]. For the remainder of this paper we follow the notation and terminology of [3] and make use of the results contained therein. In particular, an ideal $J$ in $L$ is uniformly regular provided that if $x \in J$ there is a $y \in J$ with $x \triangleleft^* y$; $RL$ denotes the frame of all uniformly regular ideals of $L$ and $k(x)$ is the uniformly regular ideal consisting of all $y \in L$ such that $y \triangleleft^* x$.

In [3] the authors establish that $RL$ is a compactification of the uniform frame $(L, U^*)$. The purpose of this section is to show that there exists a totally bounded quasi-uniformity $\overline{U}$ on $RL$ such that $\overline{U}^*$ determines $RL$ and a dense quasi-uniform frame homomorphism from $(RL, \overline{U})$ onto $(L, U)$. That is, we show that $(RL, \overline{U})$ is a compactification of the quasi-uniform frame $(L, U)$.

For each $u \in U$ define $\overline{u} : RL \rightarrow RL$ by $\overline{u}(J) = \bigvee \{ k(u(x)) : x \in S_u \text{ and } x \wedge J \neq 0 \}$, and let $\overline{B} = \{ \overline{u} : u \in U \}$. We show that $\overline{B}$ is a base for a quasi-uniformity $\overline{U}$ supported on $RL$ such that $\overline{U}^*$ determines $RL$, and such that $(RL, \overline{U})$ is a compactification of the quasi-uniform frame $(L, U)$.

In order to establish that $(RL, \overline{U})$ is a quasi-uniform frame, we need the following lemmas.

**Lemma 3.1.** Let $u \in U$. If $x$ is a $u$-small element of $L$, then $k(x)$ is $\overline{u}$-small, and if $J \in RL$ is $\overline{u}$-small and $x \in J$, then $x$ is $u^2$-small.

**Proof:** Let $x$ be a $u$-small element of $L$. Let $J \in RL$ such that $J \cap k(x) \neq \{0\}$. Let $y \in k(x)$ and let $a \in J \cap k(x)$, $a \neq 0$. Then $a \wedge x \neq 0$ and so $x \leq u(a)$. Thus $y \triangleleft^* x \leq u(a)$ and so $y \in k(u(a))$. Therefore $k(x) \subseteq k(u(a)) \subseteq \overline{u}(J)$.

Let $J$ be a $\overline{u}$-small element of $RL$ and let $x \in J$. Suppose that $y \wedge x \neq 0$. Since $x \in J$, $k(x) \subseteq J$ and since $J$ is $\overline{u}$-small, $k(x)$ is $\overline{u}$-small. Note that $k(x \wedge y) \subseteq k(x) \wedge k(y)$ and $0 \neq x \wedge y = \bigvee k(x \wedge y)$ so that $k(x) \leq \overline{u}(k(y))$. Thus $x = \bigvee k(x) \leq \bigvee \overline{u}(k(y))$.

Let $a \in \overline{u}(k(y))$. Then $a = \bigvee_{i=1}^{n} a_i$ where for $1 \leq i \leq n$ there exist $z_i$ and $q_i$ such that $a_i \triangleleft^* u(z_i)$, $z_i$ is $u$-small, $z_i \wedge q_i \neq 0$ and $q_i \triangleleft^* y$. For $1 \leq i \leq n$, $z_i \leq u(q_i) \leq u(y)$ and so $a_i \leq u(z_i) \leq u(y)$. Hence $x \leq \bigvee \overline{u}(k(y)) \leq u^2(y)$.

**Lemma 3.2.** Let $a, b \in L$ and suppose that $u \in U$ such that $u^*(b) \leq a$. Let $w \in U$ such that $w^4 \leq u$. Then $\overline{u}^*(k(b)) \subseteq k(a)$.
Proof: Let $J$ be a $\overline{\cap}$-small member of $\mathcal{RL}$ such that $J \cap k(b) \neq \{0\}$. Let $y \in J$ and $z \in J \cap k(b)$, $z \neq 0$. Then $y \lor z \in J$ and by Lemma 3.1, $y \lor z$ is $w^2$-small. Therefore by [8, Proposition 2.1], $y \leq y \lor z \leq (w^2)^* (b) <^* u^*(b) \leq a$ and so $J \subseteq k(a)$. □

Proposition 3.3. Let $U$ be a totally bounded frame quasi-uniformity and let $L = Fr(U^*)$. Let $\overline{U} = \{\overline{u} : u \in U\}$. Then $\overline{U}$ is a base for a totally bounded frame quasi-uniformity $U$ such that $(\mathcal{RL}, U)$ is a quasi-uniform frame.

Proof: Let $u \in U$, let $J \in \mathcal{RL}$ and let $a \in J$. Since $U$ is totally bounded, $a = \bigvee_{i=1}^{n} a_i$ where each $a_i \in S_u$. Thus $a = \bigvee_{i=1}^{n} \bigwedge_{i=1}^{n} k(u(a_i)) \subseteq \overline{u}(J)$. Hence $J \subseteq \overline{u}(J)$ and it is clear that $\overline{u}$ is a join homomorphism.

Let $w \in U$ and let $u \in U$ such that $u^3 \leq w$, and let $J \in \mathcal{RL}$.

$$\overline{u}(\overline{u}(J)) = \overline{u}
\left(\bigvee \left\{ k(u(c)) : c \in S_u \text{ and } c \lor \bigvee J \neq 0 \right\} \right)
\subseteq \bigvee \left\{ k(u(b)) : b, c \in S_u, b \lor \bigvee k(u(c)) \neq 0, \text{ and } c \lor \bigvee J \neq 0 \right\}
\subseteq \bigvee \left\{ k(w(c)) : c \in S_w \text{ and } c \lor \bigvee J \neq 0 \right\}
\subseteq \overline{u}(J).$$

To see that axiom (2) holds for $\overline{U}$, let $u, w \in U$ and let $J \in \mathcal{RL}$.

$$\overline{u} \land \overline{w}(J) = \bigvee \left\{ k((u \land w)(a)) : a \in S_{u \land w} \text{ and } a \land \bigvee J \neq 0 \right\}
\subseteq \bigvee \left\{ k(u(b) \land w(c)) : b, c \in S_{u \land w}, b \land \bigvee J \neq 0, \text{ and } c \land \bigvee J \neq 0 \right\}
\subseteq \bigvee \left\{ k(u(b)) : b \in S_u \text{ and } b \land \bigvee J \neq 0 \right\} \land \bigvee \left\{ k(w(c)) : c \in S_w \text{ and } c \land \bigvee J \neq 0 \right\}
= \overline{u}(J) \cap \overline{w}(J).$$

Let $u \in U$. Since $U$ is totally bounded, there is a finite subcover $A$ of $S_u$. Banaschewski and Pultr [3, p. 67] prove that $\{k(x) : x \in A\} = L$. Thus, it follows from Lemma 3.1 that for each $u \in U$, $\overline{u}$ is a $\Delta$-map and it also follows that $\overline{U}$ is totally bounded.

It remains to show that $\overline{U}^*$ determines $\mathcal{RL}$. Let $J \in \mathcal{RL}$. Then $J = \bigvee \{k(a) : k(a) \subseteq J\}$. Let $b \in J$. There exists $a \in J$ such that $b \triangleleft^* a$. By Lemma 3.2, there exists $w \in U$ such that $\overline{w}^*(k(b)) \subseteq k(a) \subseteq J$. Hence $k(b) \triangleleft^* J$. □

Proposition 3.4. The function $g : (\mathcal{RL}, \overline{U}) \to (L, U)$ defined by join is a dense quasi-uniform frame homomorphism onto $(L, U)$.

Proof: Let $a \in L$. Since $a = \bigvee \{b : b \triangleleft^* a\} = \bigvee k(a)$, $g$ maps onto $(L, U)$. Clearly $g^{-1}(0) = \{0\}$. Let $\overline{u} \in \overline{U}$ and let $v \in U$ such that $v^2 \leq u$. We show that $v \circ g \leq g \circ \overline{u}$. Let $J \in \mathcal{RL}$. Then $\overline{u}(J) = \bigvee \{k(u(a)) : a \in S_u \text{ and } a \lor \bigvee J \neq 0\}$.
and 
\[ g \circ \bar{u}(J) = \bigvee \{ V \{ k(u(a)) : a \in S_u \text{ and } a \wedge J \neq 0 \} \}. \]

On the other hand
\[ v \circ g(J) = v(\bigvee J) = \bigvee \{ v(a) : a \in S_u \text{ and } a \wedge J \neq 0 \}. \]

Since \( v(a) \triangleleft u(a) \), \( v(a) \in \bigvee \{ V \{ k(u(a)) : a \in S_u \text{ and } a \wedge J \neq 0 \} \}). \)

It follows from Theorem 3.2 that \( \bar{U}^* \) is a uniformity that determines \( \mathcal{R}L \) and it follows from [3, Corollary to Lemma 2 and Lemma 4] that \( \bar{U}^* \) is the only uniformity that determines \( \mathcal{R}L \). The join map from \( (\mathcal{R}L, \bar{U}) \) to \( (L, U) \) is the required dense quasi-uniform frame homomorphism. □

4. The bicompletion of a quasi-uniform frame.

In this final section, we consider the sense in which the Cauchy spectrum of a quasi-uniform frame, introduced by Banaschewski and Pultr [3], can be viewed as the spectrum of its bicompletion. We make use of the result [3, Proposition 9] that the Cauchy spectrum of a uniform frame \( (L, U) \) is the spectrum of its completion \( C\mathcal{L} \). In order to make this section dovetail with [3], we use covering uniformities.

For a given quasi-uniform frame \( (L, U) \) the collection of covers \( \{ S_u : u \in U \} = \{ S_u : u \in U^* \} \) generates the covering uniformity \( U \) corresponding to the entourage uniformity \( U^* \) [5]. Let \( (L, U) \) be a quasi-uniform frame. A filter \( F \) in \( L \) is a \( U \)-Cauchy filter provided that for each \( u \in U \), \( S_u \cap F \neq \emptyset \). It is shown in [3] that a \( U^* \)-Cauchy filter is \( U^* \)-regular if, and only if, it is a minimal \( U^* \)-Cauchy filter.

Given a covering uniformity \( U \), Banaschewski and Pultr construct the uniform space \( \psi L \) whose ground set is the collection of all minimal Cauchy filters and whose uniformity is generated by the covers \( \psi_A = \{ \psi_a : a \in A \} \) where \( A \in U \) and for each \( a \in A, \psi_a = \{ F \in \psi L : a \in F \} \). They call the resulting uniform space the Cauchy spectrum of the uniform frame \( (L, U) \).

We make repeated use of the following proposition.

**Proposition 4.1** [8]. Let \( (X, U) \) be a quasi-uniform space, let \( A \) and \( B \) be \( T(U) \)-open sets and let \( U \) be an open neighbour of \( X \). Let \( u : T(U) \rightarrow T(U) \) be defined by \( u(G) = U(G) \). If \( A \times B \subseteq U \), then \( A \upharpoonright B \leq u \). If \( A \upharpoonright B \leq u \), then \( A \times B \subseteq \bar{U} \), where the closure is taken either with respect to \( T(U) \times T(U) \) or with respect to \( T(U) \times T(U^{-1}) \).

**Proposition 4.2.** Let \( U \) be a frame quasi-uniformity and let \( L = Fr(U^*) \). For each \( u \in U \) set \( \bar{u} = \{ (F, G) \in \psi L \times \psi L : \text{there exist } x \in F \text{ and } y \in G \text{ such that } x \upharpoonright y \leq u \} \).

Then \( \bar{U} = \{ \bar{u} : u \in U \} \) is a base for a quasi-uniformity on \( \psi L \) and \( (\psi L, \bar{U}^*) \) is the Cauchy spectrum of \( L \).

**Proof:** We first prove that \( \bar{U} \) is a base for a quasi-uniformity on \( \psi L \). Let \( u, v \in U \). Then \( u \wedge v \in U \) and \( u \wedge v = \bar{u} \cap \bar{v} \). Moreover, for each \( F \in \psi L \) there exists a \( u \)-small \( x \in F \) and since \( x \upharpoonright x \leq u \), \( (F, F) \in \bar{u} \).

Let \( u \in U \) and let \( v \in U \) such that \( v^2 \leq u \). To show that \( \bar{v}^2 \subseteq \bar{u} \), let \( (F, G) \) and \( (G, H) \) belong to \( \bar{v} \). There are \( x \in F \) and \( y \in G \) such that \( x \upharpoonright y \leq v \) and \( p \in G \) and \( q \in H \) such that \( p \upharpoonright q \leq v \). Since \( y \wedge p \neq 0 \), \( x \upharpoonright q \leq u \). Thus \( \bar{v}^2 \subseteq \bar{u} \).

In view of [8, Proposition 2.1] and the introductory remarks of this section, in order to show that \( (\psi L, \bar{U}^*) \) is the Cauchy spectrum it suffices to prove that \( \{ S_{\bar{u}} : \bar{u} \in \bar{U} \} \) is a base for the covering uniformity given by Banaschewski and
Pultr [3]. Let \( w \in U \) and let \( z, v \in U \) such that \( v^3 \leq w \) and \( z^2 \leq v \). There exists \( \tilde{u} \in \tilde{U} \) such that \( \tilde{u} \) is closed in the topology \( \tau(\tilde{U}) \times \tau(\tilde{U}^{-1}) \) and \( \tilde{u} \subseteq \tilde{z} \) [9, page 8]. We show that \( S_{\tilde{u}} \) refines \( \psi_{S_{\tilde{u}}} \). Let \( T \in S_{\tilde{u}} \). Since \( T \) is \( \tilde{u} \)-small set of minimal \( U \)-Cauchy filters, \( T \parallel T \leq \tilde{u} \) and by Proposition 4.1, \( T \times T \subseteq \tilde{u} \). Let \( \tilde{F} \in T \). There exist \( x_1, x_2 \in F \) and \( y_1, y_2 \in G \) such that \( x_1 \parallel y_1 \leq z \) and \( y_2 \parallel x_2 \leq z \). Set \( x = x_1 \wedge x_2 \) and \( y = y_1 \wedge y_2 \) and note that \( x \neq 0, y \neq 0, y \in G, x \parallel y \leq z \) and \( y \parallel x \leq z \). By definition, \( x \parallel y \leq \tilde{z} \) and so \( y \leq z(a) \wedge \tilde{z}(a) \). It follows from [8, Lemma 3.12] that \( v^*(a) \) and so \( G \in \psi_{v^*(a)} \). By [8, Proposition 3.9(2)], \( v^*(a) \) is \( v^3 \)-small; hence \( \psi_{v^*(a)} \in \psi_{A_w} \). Thus \( S_{\tilde{u}} \) refines \( \psi_{S_{\tilde{u}}} \).

To show that \( \psi_{S_{\tilde{u}}} \subseteq S_{\tilde{u}} \), let \( a \in S_u \) and let \( F, G, \in \psi_{a} \). Then \( a \in F \cap G \) and \( a \parallel a \leq u \) so that \( (F, G) \in \tilde{u} \). Then \( \psi_{a} \times \psi_{a} \subseteq \tilde{u} \) and so by Proposition 4.1, \( \psi_{a} \in S_{\tilde{u}} \). □

It follows from Proposition 4.1 and the proof of [9, Theorem 3.33] that \((\psi L, U^*)\) is the bicompletion of \((L, U)\) whenever \( U \) is a quasi-uniformity on a set \( X \) and \( L = T(U^*) \).

**References**


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