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Some conditions under which a uniform space is fine

Umberto Marconi

Abstract. Let $X$ be a uniform space of uniform weight $\mu$. It is shown that if every open covering, of power at most $\mu$, is uniform, then $X$ is fine. Furthermore, an $\omega_\mu$-metric space is fine, provided that every finite open covering is uniform.

Keywords: uniform space, uniform weight, fine uniformity, uniformly locally finite, $\omega_\mu$-additive space, $\omega_\mu$-metric space

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0. A recurrent problem about uniform spaces is to see whether a uniformity is the finest one compatible with the topology. Isiwata and Atsuji solved this problem in metric spaces [2], [6]. The following theorem summarizes some equivalent conditions of Theorem 1 of [2].

Theorem 1. The following conditions on a metric space $X$ are equivalent:

1. every open covering is uniform;
2. every countable open covering is uniform;
3. every open covering consisting of two elements is uniform;
4. the subset $K$ of limit points is compact and, for every uniform covering $\mathcal{U}$, the subspace $X \setminus \text{St}(K, \mathcal{U})$ is uniformly discrete.

(The star $\text{St}(K, \mathcal{U})$ of $K$ with respect to $\mathcal{U}$ is the union of all elements of $\mathcal{U}$ which have a non-empty intersection with $K$).

It is interesting to see if a suitable version of Theorem 1 also holds in uniform spaces. Metric spaces are uniform spaces with a countable base for the uniformity; this base can be assumed to be well ordered by star-refinement. In uniform spaces, the existence of a well ordered base is a very strong property.

We will prove that an equivalence analogous to $1 \Leftrightarrow 2$ holds for general uniform spaces and depends only on cardinal properties, while the equivalence $1 \Leftrightarrow 3$ can be generalized to $\omega_\mu$-metric spaces (= uniform spaces which admit a base of uniform coverings well ordered by a regular cardinal $\omega_\mu$). In $\omega_\mu$-metric spaces we will provide a suitable formulation of the condition 4 (obviously in uniform spaces $3 \neq 1$ and $1 \neq 4$).

1. Unless otherwise specified, the space $X$ is a uniform space, with the uniform topology.

The following lemma is useful for working with cardinal properties of locally finite families. The proof is easy to check (for example, see [7]).
Lemma 1. Let $G$ be a locally finite family of subsets of $X$. If the power of $G$ is at most $\mu$, then there exists an open covering $\mathcal{B}$, of power at most $\mu$, such that every element of $\mathcal{B}$ meets only finitely many elements of $G$.

We recall that the uniform weight of $X$ is the smallest cardinal number of a base for the uniformity.

The following theorem is the uniform analogue of $2 \Rightarrow 1$ in Theorem 1.

Theorem 2. Let $\mu$ be the uniform weight of $X$. If every open covering of power at most $\mu$ is uniform, then every open covering is uniform.

We shall prove Theorem 2 in three steps (the statements of each step hold under the hypotheses of Theorem 2).

Let $\{U_\alpha : \alpha < \mu\}$ be a base for the uniformity, consisting of open uniform coverings.

Step 1. $X$ is a paracompact topological space.

Proof: Let $\mathcal{A}$ be an open covering. For every $x \in X$, choose $\alpha(x) < \mu$ such that $\text{St}(x, U_{\alpha(x)})$ is contained in some element of $\mathcal{A}$. For every $\alpha$, let

$$A_\alpha = \bigcup_{\alpha(x) = \alpha} \text{St}(x, U_{\alpha(x)}).$$

By assumption, the covering $\{A_\alpha : \alpha < \mu\}$ is uniform.

The covering

$$\mathcal{B} = \{A_\alpha \cap \text{St}(x, U_{\alpha(x)}) : \alpha < \mu, \alpha(x) = \alpha\}$$

is an open refinement of $\mathcal{A}$.

Since uniform coverings are normal coverings (in the sense of Tukey), by the condition $(g)$ in [9, Theorem 1.2] it follows that the covering $\mathcal{B}$ has an open star-refinement.

A family $\mathcal{F}$ of subsets of $X$ is said to be uniformly locally finite if there exists a uniform covering $\mathcal{B}$ such that every element of $\mathcal{B}$ meets $\mathcal{F}$ in only a finite number of elements.

Step 2. Every locally finite family $\mathcal{F}$ is uniformly locally finite.

Proof: We proceed by contradiction. Suppose that for every $\alpha$ there exist an element $U_\alpha \in U_\alpha$ and a countable subfamily $\mathcal{F}_\alpha$ of $\mathcal{F}$ such that $U_\alpha \cap F \neq \emptyset$ for every $F \in \mathcal{F}_\alpha$.

Let $\mathcal{G}$ be the union of all subfamilies $\mathcal{F}_\alpha$. $\mathcal{G}$ is a locally finite family of subsets and the power of $\mathcal{G}$ is at most $\mu$. By Lemma 1, there exists an open covering $\mathcal{B}$ of power at most $\mu$ such that every element of $\mathcal{B}$ meets $\mathcal{G}$ only in a finite number of elements. $\mathcal{B}$ is an open covering of power at most $\mu$, which cannot be uniform because the family $\mathcal{G}$ is not uniformly locally finite. This is a contradiction with the hypothesis of Theorem 2.

\[ \square \]
Step 3. Every locally finite open covering $\mathcal{A}$ is uniform.

Proof: By Step 2, $\mathcal{A}$ is uniformly locally finite. For every $\alpha$, by possibly refining coverings $\mathcal{U}_\alpha$, we can assume that every element of $\mathcal{U}_\alpha$ meets $\mathcal{A}$ only for a finite number of elements.

If $\mathcal{A}$ is not a uniform covering, then for every $\alpha$ there exists $U_\alpha \in \mathcal{U}_\alpha$ such that $U_\alpha \setminus A \neq \emptyset$ for every $A \in \mathcal{A}$.

Let $A_\alpha = \{ A \in \mathcal{A} : A \cap U_\alpha \neq \emptyset \}$ and let $C = \bigcup_\alpha A_\alpha$.

Every $A_\alpha$ is a finite family, thus the power of $C$ is at most $\mu$. Therefore we have a contradiction, because the open covering $C \cup \{ \bigcup(A \setminus C) \}$, of power at most $\mu$, cannot be uniform. In fact, for every $\alpha$, $U_\alpha \cap (\bigcup(A \setminus C)) = \emptyset$ and $U_\alpha \setminus A \neq \emptyset$ for every $A \in C$. \hfill \Box

The conclusion of Theorem 2 follows from Step 1 and Step 3.

Remark. One might conjecture that every open covering is uniform, provided that the open coverings of power less than $\mu$ are uniform coverings. This, however, is not the case. For a counterexample, let $X$ be the space of ordinals less than $\omega_1$, equipped with the unique (precompact) uniformity (an open covering is uniform iff it has a finite subcovering).

By countable compactness of $X$, every countable open covering is uniform and it is easy to verify that the uniform weight of $X$ is $\omega_1$. Furthermore, $X$ is not a paracompact topological space [4, p. 380].

2. Denote by $C^*$ the weak uniformity of continuous bounded real functions on a completely regular Hausdorff space $X$.

$X$ is a normal space iff every open covering consisting of two elements belongs to $C^*$.

It is an interesting question to see when a uniformity finer than $C^*$ is fine. For example, the implication $3 \Rightarrow 1$ of Theorem 1 says that metric uniformities finer than $C^*$ are fine. Another example of uniform space with this property are sequentially uniform spaces [3].

In the next theorem, we extend the equivalences $1 \Leftrightarrow 3 \Leftrightarrow 4$ of Theorem 1 to $\omega_\mu$-metric spaces. Notice that the proof of this theorem follows from ordinal properties.

An $\omega_\mu$-metric space is a uniform space which admits a base of uniform coverings

$$\mathcal{B} = \{ \mathcal{U}_\alpha : \alpha < \omega_\mu \}$$

well ordered by refinement (hence by star-refinement) by a regular cardinal $\omega_\mu$.

An $\omega_\mu$-metric space is paracompact (ultra-paracompact if $\mu > 0$) [1].

Let $\lambda$ be a cardinal number. A topological space is said to be $\lambda$-compact if every open covering has a subcovering of power less than $\lambda$. A weakly paracompact space $X$ is $\lambda$-compact iff the power of every discrete closed subset of $X$ is less than $\lambda$ (as one can prove by mimicking the proof of [4, Theorem 5.3.2]).

In the proof of Theorem 3, the base $\mathcal{B}$ is assumed well ordered by star-refinement. The equivalence $1 \Leftrightarrow 2$ has been already proved in [8].
Theorem 3. Let $X$ be an $\omega_\mu$-metric space. The following conditions are equivalent:

1. every open covering is uniform;
2. the set $K$ of limit points is $\omega_\mu$-compact and for every $\alpha$ the subspace $X \setminus \text{St}(K, U_\alpha)$ is uniformly discrete;
3. every finite open covering is uniform.

Proof: $1 \Rightarrow 3$ Obvious.

$3 \Rightarrow 2$ By way of contradiction, assume that there exists a closed discrete subset $D = \{x_\alpha : \alpha < \omega_\mu\}$ of pairwise distinct limit points.

We shall prove that, for every $\alpha$, one can choose $\beta(\alpha) \geq \alpha$ such that the collection $\mathcal{F} = \{\text{St}(x_\alpha, U_{\beta(\alpha)}) : \alpha < \omega_\mu\}$ consists of pairwise disjoint subsets. We proceed by transfinite induction. Choose $\beta(0) \geq 0$ such that $\text{St}(x_0, U_{\beta(0)})$ is disjoint from $D \setminus \{x_0\}$. Let $\alpha > 0$ and $C_\alpha = \bigcup_{\gamma < \alpha} \text{St}(x_\gamma, U_{\beta(\gamma)})$. The set $C_\alpha$ is closed, because $X$ is an $\omega_\mu$-additive topological space [1]. The set $C = C_\alpha \cup \{x_\gamma : \gamma > \alpha\}$ is closed and therefore there exists $\beta(\alpha) \geq \alpha$ such that the subset $\text{St}(x_\alpha, U_{\beta(\alpha)})$ is disjoint from $C$.

Choose $y_\alpha \in \text{St}(x_\alpha, U_{\beta(\alpha)})$, $y_\alpha \neq x_\alpha$. The subset $F = \{y_\alpha : \alpha < \omega_\mu\}$ has no limit points and the open covering $\{X \setminus F, X \setminus D\}$ cannot be uniform, because the subsets $F$ and $D$ cannot be separated by a uniform covering.

What we still need to prove is that for every $U \in \mathcal{B}$ the subspace $Y = X \setminus \text{St}(K, U)$ is uniformly discrete (notice that every subset of $Y$ is closed in $X$). We argue by way of contradiction. Again using transfinite induction, it is easy to choose elements $x_\alpha$, $y_\alpha$ such that $y_\alpha \in \text{St}(x_\alpha, U_\alpha)$ and $x_\alpha \neq y_\beta$ for every $\alpha$, $\beta$. Thus the open covering consisting of $X \setminus \{x_\alpha : \alpha < \omega_\mu\}$ and $X \setminus \{y_\alpha : \alpha < \omega_\mu\}$ cannot be uniform. This contradiction concludes the proof.

$2 \Rightarrow 1$

It is enough to prove that the trace on $K$ of every open covering is a uniform covering of $K$. This trace can be refined by a covering of the form $\{\text{St}(x, U_{\alpha(x)}) : x \in K\}$. As $K$ is $\omega_\mu$-compact, the covering $\{\text{St}(x, U_{\alpha(x)}) : x \in K\}$ has a subcovering of power less than $\omega_\mu$, say $\{\text{St}(x_i, U_{\alpha(x_i)}) : i < \delta\}$ for a suitable $\delta < \omega_\mu$. Thus $\{\text{St}(x, U_{\gamma}) : x \in K\}$ where $\gamma = \sup\{\alpha(x_i) : i < \delta\}$ is the required uniform refinement (see [4, Theorem 4.3.31]). ⊓⊔

Remark. It follows from the proof that the condition 2 of the above theorem can be strengthened as follows:

2′ the set $K$ of limit points is $\omega_\mu$-compact and every closed discrete subset of $X$ is uniformly discrete.

Remark. It is well-known that the fine uniformity on a metrizable topological space $X$ is a metric uniformity iff the set of limit points is compact (see for example [10]).

We can see that an analogous result holds for $\omega_\mu$-metrizable spaces: precisely, if the subset $K$ of limit points of an $\omega_\mu$-metrizable space $X$ is $\omega_\mu$-compact, then the fine uniformity is an $\omega_\mu$-metric uniformity.
Let \[ B = \{ \mathcal{U}_\alpha : \alpha < \omega_\mu \} \]
be a well ordered base for a compatible uniformity. For every \( \alpha \), consider the open covering
\[ \mathcal{V}_\alpha = \{ \{x\}, U : U \in \mathcal{U}_\alpha, U \cap K \neq \emptyset, x \in X \setminus \text{St}(K, \mathcal{U}_\alpha) \} . \]

It is easy to check that \( C = \{ \mathcal{V}_\alpha : \alpha < \omega_\mu \} \) is a well ordered (by refinement) base which induces the fine uniformity (see Theorem 3).

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