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On subspaces of pseudo-radial spaces

JIN-YUAN ZHOU

Abstract. It is proved that, under the Martin’s Axiom, every $T_1$-space with countable tightness is a subspace of some pseudo-radial space. We also give several characterizations of subspaces of pseudo-radial spaces and conclude that being a subspace of a pseudo-radial space is a local property.

Keywords: pseudo-radial spaces, prime spaces, sub pseudo-radial spaces, tightness, Martin’s Axiom

Classification: 54A35, 54B05, 54D99

1. Introduction.

In [1] the authors proposed the following problem: find necessary or sufficient (or both) conditions for a topological space to be a subspace of a pseudo-radial space. They also asked whether, in particular, $N \cup \{p\}$ is a subspace of a pseudo-radial space for $p \in \beta N \setminus N$. In Section 2 we give some necessary and sufficient conditions for a space to be a subspace of a pseudo-radial space. In Section 3 we prove that, under Martin’s Axiom, every $T_1$ space with countable tightness is a subspace of a pseudo-radial space. Thus we partly answer the question 3.4 of [1].

Definition 1.1. A subset $A$ of a topological space $X$ is called closed w.r.t. chain-net if for each $x \in X$, if there exists a transfinite sequence in $A$ converging to $x$, then $x \in A$. For any $B \subseteq X$ we denote by $\text{clseq}_X B$ the smallest subset of $X$ containing $B$ and closed w.r.t. chain-net.

Definition 1.2 (5). A space is called pseudo-radial if for each $A \subseteq X$, $\overline{A} = \text{clseq}_X A$. A space is called sub pseudo-radial if it is a subspace of some pseudo-radial space.

There was a lot of equivalent definitions of pseudo-radial spaces (see [1] and [2]). All spaces are assumed to be $T_1$. If $\{X_\alpha : \alpha \in \Sigma\}$ is a family of spaces, we denote by $\bigoplus_{\alpha \in \Sigma} X_\alpha$ the topological sum of $\{X_\alpha : \alpha \in \Sigma\}$.

2. Some characterizations.

We start with a lemma.

Lemma 2.1. Every quotient of a sub pseudo-radial space is sub pseudo-radial.

Proof: Since every quotient of a pseudo-radial space is pseudo-radial, it is enough to see that for any class $M$ of spaces, if $M$ is closed under quotient mappings, then the class consisting of subspaces of the spaces in $M$ is also closed under quotient mappings. \qed
We call a space a prime space if it has only one non-isolated point. Given any space \( X \) and a point \( p \) in \( X \), denote by \( X_p \) the prime space constructed by making each point, other than \( p \), isolated with \( p \) retaining its original neighborhoods. We call \( X_p \) the prime factor of \( X \) at \( p \). Obviously, each topological space is the quotient of the topological sum of all its prime factors.

**Proposition 2.2.** For a space \( X \) the following conditions are equivalent:

(i) \( X \) is sub pseudo-radial,

(ii) for every \( p \) in \( X \), \( X_p \) is sub pseudo-radial,

(iii) for each subset \( A \) of \( X \) and \( q \in \overline{A} \), there exists a subset \( B \) of \( A \) such that \( q \in \overline{B} \) and \( B \cup \{q\} \) is sub pseudo-radial.

**Proof:** The implication (i) \( \rightarrow \) (iii) is obvious. The proof of the implication (i) \( \rightarrow \) (ii) is completely the same as that of Proposition 5.1 of [3].

To prove the left two implications, let \( Z = \bigoplus_{p \in X} X_p \) when (ii) holds and \( Z = \bigoplus\{Y : Y \subseteq X \text{ and } Y \text{ is sub pseudo-radial}\} \) when (iii) holds. It is easy to see that, in both cases, \( X \) is a quotient of \( Z \) and \( Z \) is sub pseudo-radial. By virtue of Lemma 2.1, \( X \) is a pseudo-radial space when (ii) or (iii) holds. \( \square \)

**Corollary 2.3.** A space \( X \) is sub pseudo-radial if either

(i) each subset of \( X \) with cardinality not greater than the tightness of \( X \) is sub pseudo-radial, or

(ii) each point of \( X \) has a sub pseudo-radial neighborhood.

## 3. Countable case.

In this section, \( N \) denotes the set of natural numbers. \( \beta N \) is the Čech-Stone compactification of the discrete space \( N \). If \( A \) and \( B \) are subsets of \( N \), \( A \subseteq \ast B \) means that there exists an \( n \) in \( N \) such that \( A \setminus \{0, 1, 2, \ldots, n - 1\} \subseteq B \). A family \( A \) of subsets of \( N \) is called an almost disjoint family, shortened as a.d. family, if for any distinct elements \( A_1 \) and \( A_2 \) of \( A \), \( A_1 \cap A_2 \) is finite. We say that \( A \) has sfip (strong finite intersection property) if every nonempty finite subfamily of \( A \) has infinite intersection. We say that \( B \) is a pseudo-intersection of \( A \) if \( B \subseteq \ast A \) for each \( A \) in \( A \). For any set \( A \), \( |A| \) denotes the cardinality of \( A \); \( c \) denotes the cardinality of the power set \( \mathcal{P}N \) of \( N \).

The following lemma is well-known in set-theory (for example, see 11C of [14]).

**Lemma 3.1** (MA). For each family \( A \) of subsets of \( N \), if \( |A| < c \) and \( A \) has sfip, then \( A \) has an infinite pseudo-intersection.

**Theorem 3.2** (MA). Every space with countable tightness is sub pseudo-radial.

**Proof:** It is a consequence of the following Theorem 3.3 and (i) of Corollary 2.3, \( \square \)

**Theorem 3.3** (MA). Every countable space is sub pseudo-radial.

**Proof:** By virtue of Proposition 2.2, we only need to prove that every countable prime space is sub pseudo-radial. Let \( X = N \cup \{p\} \) be a prime space with the unique non-isolated point \( p \). We prove the \( X \) is sub pseudo-radial.
W.l.o.g., we assume that $\chi(p, X) = c$. Let $\mathcal{B}$ be a filter base on $N$ such that the set $\{B \cup \{p\} : B \in \mathcal{B}\}$ constitutes a local base at $p$. Let $\mathcal{A} = \mathcal{B} \cup \{A \subseteq N; \ p \in \overline{A}^X$ and $A$ contains no infinite pseudo-intersection of $B\}$.

Let $\mathcal{A} = \{A_\alpha : \alpha < c\}$ be an enumeration of $\mathcal{A}$ such that for each $A \in \mathcal{A}$, the set $\{\alpha < c : A_\alpha = A\}$ is unbounded in $c$. We construct by induction an almost disjoint sequence $\mathcal{C} = \{C_\alpha : \alpha < c\}$ and a sequence $\{B_\alpha : \alpha < c\} \subseteq \mathcal{B}$ such that

(i) $\forall \alpha < c$, $C_\alpha \subseteq A_\alpha$ and $C_\alpha$ is infinite;
(ii) $\forall \beta < \alpha < c$, if $A_\beta \in \mathcal{B}$, then $C_\alpha \subseteq *A_\beta$;
(iii) $\forall \alpha < c$, $C_\alpha \cap B_\alpha = \emptyset$.

Assume $\alpha < c$ and we have constructed $\{C_\beta : \beta < \alpha\}$ and $\{B_\beta : \beta < \alpha\}$ satisfying (i) to (iii). We construct $C_\alpha$, $B_\alpha$ as follows.

**Case I.** $A_\alpha \notin \mathcal{B}$. Since $p \in \overline{A_\alpha}^X$, we apply Lemma 3.1 on the family

$$\mathcal{B}' = \{B_\beta \cap A_\alpha : \beta < \alpha\} \cup \{A_\beta \cap A_\alpha : \beta \leq \alpha \text{ and } A_\beta \in \mathcal{B}\}.$$  

We obtain an infinite subset $A$ of $A_\alpha$ which is a pseudo-intersection of $\mathcal{B}'$. Since $A$ cannot be a pseudo-intersection of $\mathcal{B}$, there is a $B \in \mathcal{B}$ such that $A \setminus B$ is infinite. Let $C_\alpha = A \setminus B$ and $B_\alpha = B$.

**Case II.** $A_\alpha \in \mathcal{B}$. Let $\mathcal{B}'$ as in the Case I. Since $X$ is a $T_1$ space and $|\mathcal{B}'| < c = \chi(p, X)$, there exists a $B^* \in \mathcal{B}$ such that for each finite subfamily $\mathcal{B}'$ of $\mathcal{B}$, $\bigcap_{B \in \mathcal{B}'} B \setminus B^*$ is infinite. Therefore the family $\mathcal{F} = \{B \setminus B^* : B \in \mathcal{B}\}$ has the sfp. Again by Lemma 3.1, we obtain an infinite $A \subseteq A_\alpha \setminus B^*$ which is a pseudo-intersection of $\mathcal{B}'$. Let $C_\alpha = A$ and $B_\alpha = B^*$. Thus we have finished the induction.

Now we construct a Hausdorff pseudo-radial space $Y$ containing $X$ as a subspace. Let $Y = X \cup (c \times \{0\})$. We define a topology on $Y$ as follows. The set $N$ is open discrete in $Y$. For each $\alpha < c$, let $\{C_\alpha \setminus n \cup \{(\alpha, 0)\} : n \in N\}$ be a local base at the point $(\alpha, 0)$. For the point $p$, let $\{U(A_\alpha) : A_\alpha \in \mathcal{B}, \ \alpha < c\}$ be a local base, where $U(A_\alpha) = \{p\} \cup A_\alpha \cup \{(\beta, 0) : \alpha < \beta < c\}$. It is easy to see that the above topology is well-defined and that $X$ is a subspace of $Y$. $Y$ is Hausdorff because of the above property (iii) and the fact that, for each $B_\alpha$, the set $\{\beta < c : A_\beta = B_\alpha\}$ is unbounded in $c$. We are left to check that $Y$ is pseudo-radial. Let $E \subseteq Y$ and $y \in \overline{E}^Y$. To avoid the trivialities, we assume $y = p$ and $E \subseteq N$. Then $p \in \overline{E}^X$. If $E \notin \mathcal{A}$, then $\{(\alpha, 0) : \alpha < c \text{ and } A_\alpha = E\} \subseteq \text{clseq}_Y E$. Since the set $\{\alpha < c : A_\alpha = E\}$ is unbounded in $c$, $p \in \text{clseq}_Y \{(\alpha, 0) : A_\alpha = E\}$. Thus $p \in \text{clseq}_Y E$. If $E \notin \mathcal{A}$, then there exists an infinite subset $E'$ of $E$ which is a pseudo-intersection of $\mathcal{B}$. But this obviously implies that $p \in \text{clseq}_X E$. Therefore $p \in \text{clseq}_E$. We are done.

**Remark.** For any $p \in \beta N \setminus N$, it is easy to see that $N \cup \{p\}$ is not pseudo-radial. But by Theorem 3.2, we see that it is sub pseudo-radial under the Martin’s Axiom. Thus we partly answer the question 4 of [1].

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