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Commentationes Mathematicae Universitatis Carolinae, Vol. 35 (1994), No. 1, 9--19

Persistent URL: <http://dml.cz/dmlcz/118636>

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GHASSAN ALHERK, HENRYK HUDZIK

Abstract. Criteria in order that a Musielak-Orlicz sequence space l^Φ contains an isomorphic as well as an isomorphically isometric copy of l^1 are given. Moreover, it is proved that if $\Phi = (\Phi_i)$, where Φ_i are defined on a Banach space, X does not satisfy the δ_2^0 -condition, then the Musielak-Orlicz sequence space $l^\Phi(X)$ of X -valued sequences contains an almost isometric copy of c_0 . In the case of $X = \mathbb{R}$ it is proved also that if l^Φ contains an isomorphic copy of c_0 , then Φ does not satisfy the δ_2^0 -condition. These results extend some results of [A] and [H2] to Musielak-Orlicz sequence spaces.

Keywords: Musielak-Orlicz sequence space, copy of l^1 , copy of c_0

Classification: 46B20, 46B25, 46E30

0. Introduction

Two Banach spaces X, Y are said to be $(1 + \varepsilon)$ -isometric provided there exists a linear isomorphism $P : X \xrightarrow{\text{onto}} Y$ such that $\|P\| \|P^{-1}\| \leq 1 + \varepsilon$. It is easy to see that P is a $(1 + \varepsilon)$ -isometry if

$$\|x\|_X \leq \|Px\|_Y \leq (1 + \varepsilon)\|x\|_X$$

for any $x \in X$. We say a Banach space X contains an almost isometric (isomorphic) copy of Y if for any $\varepsilon > 0$ (for some $\varepsilon > 0$) there exists a subspace Z in X such that Z, Y are $(1 + \varepsilon)$ -isomorphic. We say a Banach space X contains an isomorphically isometric (shortly isometric) copy of Y if there exist a subspace Z of X and a linear isomorphism P from Z onto Y such $\|Px\|_Y = \|x\|_X$ for any $x \in Z$.

In the sequel X denotes a real Banach space and $\mathbb{N}, \mathbb{R}, \mathbb{R}_+$ and \mathbb{R}_+^c stand for the set of natural numbers, the set of reals, the set of nonnegative reals, and for $\mathbb{R}_+ \cup +\infty$, respectively. A map $\Phi : X \rightarrow \mathbb{R}_+^c$ is said to be an Orlicz function if it is convex, even, vanishing and continuous at 0, lower semicontinuous on the whole X and

$$(*) \quad \inf\{\Phi(x) : \|x\| = r\} \rightarrow \infty \quad \text{as} \quad r \rightarrow \infty.$$

We define a Musielak-Orlicz function Φ to be a sequence (Φ_i) of Orlicz functions (we write then $\Phi = (\Phi_i)$). Given a Banach space X , we denote by $l^\Phi(X)$ the real space of all X -valued sequences $x = (x_n)$. We write shortly l^Φ instead of $l^\Phi(\mathbb{R})$.

Given an arbitrary Musielak-Orlicz function $\Phi = (\Phi_i)$ we define a functional $I_\Phi : l^o(X) \rightarrow \mathbb{R}_+^e$ by

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x_i),$$

which is even and convex, $I_\Phi(0) = 0$ and for any $x \in l^o(X)$ the condition $I_\Phi(\lambda x) = 0$ for any $\lambda > 0$ yields $x = 0$.

Musielak-Orlicz space $l^\Phi(X)$ generated by a Musielak-Orlicz function Φ is defined as the set of all $x \in l^o(X)$ such that $I_\Phi(\lambda x) < \infty$ for some $\lambda > 0$ (cf. [T] and in the scalar case also [KR], [L], [M] and [RR]).

The subspace $h^\Phi(X)$ of $l^\Phi(X)$ is defined to be the closure in $l^\Phi(X)$ of the space $h(X)$ of all x in $l^o(X)$ which have only finite number of coordinates different from 0. The space $l^\Phi(X)$ can be equipped with the norm

$$\|x\|_\Phi = \inf\{\varepsilon > 0 : I_\Phi(x/\varepsilon) \leq 1\},$$

called the Luxemburg norm (cf. [M] and in the case of Orlicz spaces also [KR], [L] and [RR]). The space $h^\Phi(X)$ will be considered with the norm $\|\cdot\|_\Phi$ induced from $l^\Phi(X)$. $l^\Phi(X)$ and $h^\Phi(X)$ equipped with the norm $\|\cdot\|_\Phi$ are Banach spaces (cf. [T]).

We say that a Musielak-Orlicz function $\Phi = (\Phi_i)$ satisfies the δ_2^o -condition (we write $\Phi \in \delta_2^o$) if there are positive constants k and a , a sequence (c_i) with $c_i \in \mathbb{R}_+^e$ such that $\sum_{i=j}^{\infty} c_i < \infty$ for some $j \in \mathbb{N}$ and

$$\Phi_i(2x) \leq \Phi_i(x) + c_i$$

for any $i \in \mathbb{N}$ and $x \in X$ satisfying $\Phi_i(x) \leq a$.

If $\Phi = (\Phi_i)$ satisfies the δ_2^o -condition with $j = 1$ we say that Φ satisfies the δ_2 -condition. Of course, for any Musielak-Orlicz function $\Phi = (\Phi_i)$ with finite-valued Φ_i for any $i \in \mathbb{N}$ the δ_2 -condition is equivalent to the δ_2^o -condition (cf. [DH] and [K]).

Let us define for any Musielak-Orlicz function $\Phi = (\Phi_i)$ the sequence $\lambda = (\lambda_i)$ in \mathbb{R}_+ , where

$$\lambda_i = \sup\{u \in \mathbb{R}_+ : \Phi_i \text{ is linear on } [0, u] \text{ and } \Phi_i(u) \leq 1\}$$

for $i = 1, 2, \dots$.

1. Results

We start with the following theorem:

Theorem 1. *Let $\Phi = (\Phi_i)$ be a Musielak-Orlicz function with finite-valued Φ_i defined on \mathbb{R} for any $i \in \mathbb{N}$. Then $l^\Phi = (l^\Phi, \|\cdot\|_\Phi)$ contains an isometric copy of l^1 if and only if:*

- (i) Φ does not satisfy the δ_2 -condition.
- (ii) $\sum_{i=1}^{\infty} \Phi_i(\lambda_i) = \infty$.

PROOF: Sufficiency. Under our assumptions concerning Φ , the conditions δ_2 and δ_2^o are equivalent. Therefore, if Φ satisfies condition (i), then $\Phi \notin \delta_2^o$. This yields that l^Φ contains an isometric copy of l^∞ (cf. [K]) and so also an isometric copy of l^1 .

Assume now that Φ satisfies the condition (ii). Define i_1 to be the largest natural number satisfying

$$\sum_{i=1}^{i_1} \Phi_i(\lambda_i) \leq 1.$$

Then

$$\sum_{i=1}^{i_1+1} \Phi_i(\lambda_i) > 1.$$

There is a number $\alpha_i \in [0, \lambda_i)$ such that

$$\sum_{i=1}^{i_1} \Phi_i(\lambda_i) + \Phi_{i_1+1}(\alpha_1) = 1.$$

We have

$$\sum_{i=i_1+2}^{\infty} \Phi_i(\lambda_i) = \infty.$$

Define $i_2 \geq i_1 + 2$ to be the largest natural number such that

$$\sum_{i=i_1+2}^{i_2} \Phi_i(\lambda_i) \leq 1.$$

Then

$$\sum_{i=i_1+2}^{i_2+1} \Phi_i(\lambda_i) > 1.$$

There is a number $\alpha_2 \in [0, \lambda_{i_2+1})$ such that

$$\sum_{i=i_1+2}^{i_2} \Phi_i(\lambda_i) + \Phi_{i_2+1}(\alpha_2) = 1.$$

Proceeding in such a way by induction we find sequences (i_k) of natural numbers and (α_k) of numbers from the intervals $[0, \lambda_{i_k+1})$ such that

$$\sum_{i=i_{k-1}+2}^{i_k} \Phi_i(\lambda_i) + \Phi_{i_k+1}(\alpha_k) = 1$$

for $k = 1, 2, \dots$, where $i_0 = -1$ by definition. Denote

$$A_k = \{i_{k-1} + 1, \dots, i_k, i_k + 1\}, \quad k = 1, 2, \dots$$

The sets A_k are pairwise disjoint and $\bigcup_{k=1}^{\infty} A_k = \mathbb{N}$. Define a new sequence $d = (d_i)$ by

$$d_i = \begin{cases} \lambda_i & \text{if } i \in A_k \setminus \{i_k + 1\} \\ \alpha_k & \text{if } i = i_k + 1 \end{cases}$$

for $k = 1, 2, \dots$. Define now

$$f_k = \sum_{i \in A_k} d_i e_i,$$

where e_i is the i -th unit vector, i.e. $e_i = (0, \dots, 0, 1, 0, \dots)$ with 1 on the i -th place. We have

$$I_{\Phi}(f_k) = 1 \quad \text{for } k = 1, 2, \dots$$

We have also $f_k \perp f_l$ (i.e. the sequences f_k and f_l have disjoint supports) if $k \neq l$. Moreover, the coordinates of $f_k (k = 1, 2, \dots)$ belong to the intervals on which the respective Orlicz functions Φ_i are linear. Define an operator P from l^1 into l^{Φ} by the formula

$$Px = \sum_{k=1}^{\infty} x_k f_k \quad (\forall x = (x_k) \in l^1).$$

It is obvious that P is linear. Moreover

$$\begin{aligned} I_{\Phi} \left(\frac{Px}{\|x\|_{l^1}} \right) &= \sum_{k=1}^{\infty} I_{\Phi} \left(\frac{x_k f_k}{\|x\|_{l^1}} \right) \\ &= \sum_{k=1}^{\infty} \frac{|x_k|}{\|x\|_{l^1}} I_{\Phi}(f_k) = \sum_{k=1}^{\infty} \frac{|x_k|}{\|x\|_{l^1}} = 1. \end{aligned}$$

Hence

$$\left\| \frac{Px}{\|x\|_{l^1}} \right\|_{\Phi} = 1, \text{ i.e. } \|Px\|_{\Phi} = \|x\|_{l^1}.$$

This means that P is an isometry between l^1 and a closed subspace $P(l^1)$ of l^{Φ} .

Necessity. Assume that Φ satisfies the δ_2 -condition and $\sum_{i=1}^{\infty} (\lambda_i) < +\infty$. Then there is $n \in \mathbb{N}$, $n \geq 2$, such that $\sum_{i=1}^{\infty} \Phi_i(\lambda_i) \leq n$. We will prove that l^{Φ} is non- l_n^1 , i.e. for any elements x^1, \dots, x^n from the unit sphere $\mathcal{S}(l^{\Phi})$ of l^{Φ} there holds

$$\left\| \frac{1}{n} (x^1 \pm x^2 \pm \dots \pm x^n) \right\| < 1$$

for some choice of signs, which yields that l^1 can not be isometrically embedded into l^Φ .

Take arbitrary $x^1, \dots, x^n \in \mathcal{S}(l^\Phi)$. Then $I_\Phi(x^1) = \dots = I_\Phi(x^k) = 1$ for $i = 1, 2, \dots, n$. Define the set

$$A = \{i \in \mathbb{N} : \sum_{k=1}^n \Phi_i(x_i^k) > \Phi_i(\lambda_i)\}.$$

We will prove that for any $i \in A$

$$(1) \quad \Phi_i \left(\frac{(x_i^1 \pm \dots \pm x_i^n)}{n} \right) < \frac{1}{n} \sum_{k=1}^n \Phi_i(x_i^k)$$

for some choice of signs ± 1 , dividing the proof into two cases.

I. $\max\{|x_i^k| : k = 1, \dots, n\} \leq \lambda_i$ and $i \in A$. Then at least two numbers among $\Phi_i(x_i^k), k = 1, \dots, n$, must be positive. Assume that this is not true, i.e. that there is only one positive number $\Phi_i(x_i^j)$ among these numbers. Then

$$\sum_{k=1}^n \Phi_i(x_i^k) = \Phi_i(x_i^j) \leq \Phi_i(\lambda_i),$$

which contradicts the fact that $i \in A$. Therefore,

$$\max\{\Phi_i(x_i^k) : k = 1, \dots, n\} < \sum_{k=1}^n \Phi_i(x_i^k).$$

It is evident that

$$(2) \quad |x_i^1 \pm \dots \pm x_i^n| \leq \max\{|x_i^k| : k = 1, \dots, n\}$$

for some choice of signs ± 1 , whence

$$\begin{aligned} \Phi_i \left(\frac{1}{n} (x_i^1 \pm \dots \pm x_i^n) \right) &\leq \Phi \left(\frac{1}{n} \max_k |x_i^k| \right) \\ &= \frac{1}{n} \max_k \Phi_i(x_i^k) < \frac{1}{n} \sum_{k=1}^n \Phi_i(x_i^k). \end{aligned}$$

This means that inequality (1) holds true in case I.

II. $i \in A$ and $\max\{|x_i^k| : k = 1, \dots, n\} > \lambda_i$. Applying (2), we get for a choice of signs ± 1

$$\begin{aligned} \Phi_i \left(\frac{1}{n} (x_i^1 \pm \dots \pm x_i^n) \right) &\leq \Phi_i \left(\frac{1}{n} \max_k |x_i^k| \right) < \frac{1}{n} \Phi_i(\max_k |x_i^k|) \\ &= \frac{1}{n} \max_k \Phi_i(x_i^k) \leq \frac{1}{n} \sum_{k=1}^n \Phi_i(x_i^k). \end{aligned}$$

Combining both cases I and II we get inequality (1) for some choice of signs ± 1 . For the remaining $2^{n-1} - 1$ choices of signs ± 1 , we have by the convexity of Φ ,

$$(3) \quad \Phi_i \left(\frac{1}{n}(x_i^1 \pm \dots \pm x_i^n) \right) \leq \frac{1}{n} \sum_{k=1}^n \Phi_i(x_i^k) \quad (\forall i \in A).$$

Combining (1) and (3), we get

$$\sum_{\pm 1} \Phi_i \left(\frac{1}{n}(x_i^1 \pm \dots \pm x_i^n) \right) < \frac{2^{n-1}}{n} \sum_{k=1}^n \Phi_i(x_i^k) \quad (\forall i \in A).$$

Summing up both-side of the last inequality over $i \in A$, we have

$$\sum_{\pm 1} I_\Phi \left(\frac{1}{n}(x^1 \pm \dots \pm x^n) \chi_A \right) < \frac{2^{n-1}}{n} \sum_{k=1}^n I_\Phi(x^k \chi_A),$$

where χ_A denotes the characteristic function of A .

Hence it follows that

$$\begin{aligned} & 2^{n-1} - \sum_{\pm 1} I_\Phi \left(\frac{1}{n}(x^1 \pm \dots \pm x^n) \right) \\ &= \frac{2^{n-1}}{n} \sum_{k=1}^n I_\Phi(x^k) - \sum_{\pm 1} I_\Phi \left(\frac{1}{n}(x^1 \pm \dots \pm x^n) \right) \\ &\geq \frac{2^{n-1}}{n} \sum_{k=1}^n I_\Phi(x^k \chi_A) - \sum_{\pm 1} I_\Phi \left(\frac{1}{n}(x^1 \pm \dots \pm x^n) \chi_A \right) > 0, \end{aligned}$$

i.e.

$$\sum_{\pm 1} I_\Phi \left(\frac{1}{n}(x^1 \pm \dots \pm x^n) \right) < 2^{n-1}.$$

Therefore

$$I_\Phi \left(\frac{1}{n}(x^1 \pm \dots \pm x^n) \right) < 1$$

for at least one choice of signs ± 1 . Since Φ_i are finite-valued by the assumption and $\Phi \in \delta_2$, we get

$$\left\| \frac{1}{n}(x^1 \pm \dots \pm x^n) \right\| < 1$$

for at least one choice of signs (cf. [DH] and [K]), i.e. l^Φ is non- l_n^1 . This means that l^1 cannot be embedded isometrically into l^Φ , and the proof is finished. \square

Theorem 2. *Let $\Phi = (\Phi_i)$ be an arbitrary Musielak-Orlicz function defined on \mathbb{R} . Then $l^\Phi = (l^\Phi, \|\cdot\|_\Phi)$ contains an isomorphic copy of l^1 if and only if Φ or Φ^* (= the complementary function to Φ in the sense of Young) does not satisfy the δ_2^o -condition, i.e. if and only if l^Φ is not reflexive.*

PROOF: Sufficiency. If $\Phi \notin \delta_2^o$, then l^Φ contains an isometric copy of l^∞ (cf. [K]) and so of l^1 as well. Assume now that $\Phi \in \delta_2^o$ but $\Phi^* \notin \delta_2^o$. Then the dual of l^Φ is isomorphically isometric to l^{Φ^*} equipped with the Orlicz norm $\|\cdot\|_{\Phi^*}$ (cf. [M] and in the case of Orlicz spaces also [KR], [L] and [RR]). Therefore l^{Φ^*} contains an isomorphic copy of l^∞ (cf. again [K]), whence it follows that l^Φ contains an isomorphic copy of l^1 .

Necessity. Assume that both functions Φ and Φ^* satisfy the δ_2^o -condition. Then l^Φ is reflexive and so l^1 can not be embedded isomorphically into l^Φ as a nonreflexive space. \square

Theorem 3. *Let X be an arbitrary Banach space and $\Phi = (\Phi_i)$ be a Musielak-Orlicz function defined on X . Then:*

- (i) *if Φ does not satisfy the δ_2^o -condition, then $h^\Phi(X) = (h^\Phi(X), \|\cdot\|_\Phi)$ contains an almost isometric copy of c_o ;*
- (ii) *if $X = \mathbb{R}$ and $(h^\Phi = h^\Phi, \|\cdot\|_\Phi)$ contains an isomorphic copy of c_o , then Φ does not satisfy the δ_2^o -condition.*

PROOF: (i). Let

$$c_i^{k,\varepsilon} = \sup\{\Phi_i((1+\varepsilon)x) - 2^{k+1}\Phi_i(x) : \Phi_i(x) \leq 2^{-k-1}\} (\forall i, k \in \mathbb{N}, \varepsilon > 0).$$

We have that $\Phi \in \delta_2^o$ if and only if there exists $\varepsilon > 0$ and $m, k \in \mathbb{N}$ such that

$$\sum_{i=m}^{\infty} c_i^{k,\varepsilon} < \infty \quad (\text{cf. [DH] and [H1]}).$$

Define

$$\alpha_i^{k,\varepsilon} = \sup\{\Phi_i((1+\varepsilon)x) : \Phi_i(x) \leq 2^{-k-1} \quad \text{and} \quad \Phi_i((1+\varepsilon)x) - 2^{k+1}\Phi_i(x) \geq 0\}.$$

Since $\Phi_i(0) = 0 < 2^{-k-1}$, so 0 belongs to the set of these x over which the supremum in the definition of $c_i^{k,\varepsilon}$ is taken. Moreover,

$$\Phi((1+\varepsilon)0) - 2^{k+1}\Phi(0) = 0.$$

Hence it follows that $c_i^{k,\varepsilon} \geq 0$. Therefore, we can restrict ourselves in the definition of $c_i^{k,\varepsilon}$ to these x for which $\Phi_i((1+\varepsilon)x) - 2^{k+1}\Phi_i(x) \geq 0$. Hence we get $c_i^{k,\varepsilon} \leq d_i^{k,\varepsilon}$ for every $i, k \in \mathbb{N}$ and $\varepsilon > 0$. We have by the assumption that $\Phi \notin \delta_2^o$. Hence it follows that

$$\sum_{i=m}^{\infty} d_i^{k,\varepsilon} = \infty \quad (\forall m, k \in \mathbb{N}, \varepsilon > 0),$$

so we have among others $\sum_{i=1}^{\infty} d_i^{k,\varepsilon} = \infty$. Define i_1 as the largest natural number such that

$$\sum_{i=1}^{i_1} d_i^{k,\varepsilon} \leq 1,$$

whenever $d_1^{k,\varepsilon} \leq 1$ and $i_1 = 0$ whenever $d_1^{k,\varepsilon} > 1$. Then

$$\sum_{i=1}^{i_1+1} d_i^{k,\varepsilon} > 1.$$

Define in the first step $N_1 = \{1, \dots, i_1 + 1\}$. Define i_2 as the largest natural number such that

$$\sum_{i=i_1+2}^{i_2} d_i^{k,\varepsilon} \leq 1$$

if $d_{i_1+2}^{k,\varepsilon} \leq 1$ and $i_2 = i_1 + 2$ if $d_{i_1+2}^{k,\varepsilon} > 1$.

Then

$$\sum_{i=i_1+2}^{i_2+1} d_i^{k,\varepsilon} > 1.$$

Put $N_2 = \{i_1 + 2, \dots, i_2 + 1\}$. Proceeding in such a way by induction we can find a sequence (i_k) of nonnegative integers such that the sequence of pairwise disjoint sets (N_k) in \mathbb{N} defined by

$$N_k = \{i_{k-1} + 2, \dots, i_k + 1\}, i_0 := -1 \quad (k = 1, 2, \dots)$$

satisfies

$$(4) \quad \sum_{i \in N_k \setminus \{i_k + 1\}} d_i^{k,\varepsilon} \leq 1,$$

$$(5) \quad \sum_{i \in N_k} d_i^{k,\varepsilon} > 1.$$

In view of the definition of $d_i^{k,\varepsilon}$ and inequality (5), it follows that for any $\varepsilon > 0$ and $k \in \mathbb{N}$ there are $x_i \in X (x \in N_k)$ such that

$$(6) \quad \sum_{i \in N_k} \Phi_i((1 + \varepsilon)x_i) > 1,$$

$$(7) \quad \Phi_i(x_i) \leq 2^{-k-1} \quad \text{and} \quad \Phi_i((1 + \varepsilon)x_i) \geq 2^{k+1}\Phi_i(x_i) \quad (\forall i \in N_k).$$

Applying (6) and (7) we get

$$(8) \quad \sum_{i \in N_k} \Phi_i(x_i) \leq 2^{-k-1} \sum_{i \in N_k \setminus \{i_k+1\}} d_i^{k,\varepsilon} + 2^{-k-1} \leq 2^{-k-1} + 2^{-k-1} = 2^{-k}.$$

Define $y_k = \sum_{i=N_k} x_i e_i$. Then y_k have pairwise disjoint supports. In virtue of (6) and (8) we have

$$(9) \quad I_\Phi(y_k) = \sum_{i \in N_k} \Phi_i(x_i) \leq 2^{-k} \quad (\forall k \in \mathbb{N}),$$

$$(10) \quad I_\Phi((1+\varepsilon)y_k) = \sum_{i \in N_k} \Phi_i((1+\varepsilon)x_i) > 1 \quad (\forall k \in \mathbb{N}).$$

Define now an operator $P_1 : c_o \rightarrow h^\Phi(X)$ by

$$P_1 u = \sum_{k=1}^{\infty} u_k y_k \quad (\forall u = (u_k) \in c_o).$$

It is obvious that P_1 is linear. We will prove now that $P_1 u \in h^\Phi(X)$ for any $u \in c_o$. We need to prove that there is a sequence (y^l) in $l^o(X)$ such that y^l has only finite number of coordinates different from zero and $\|P_1 u - y^l\|_\Phi \rightarrow 0$ as $l \rightarrow \infty$, i.e.

$$(11) \quad I_\Phi(\lambda(P_1 u - y^l)) \rightarrow 0 \quad \text{as } l \rightarrow \infty \quad (\forall \lambda > 0).$$

Take arbitrary $\lambda, \varepsilon > 0$ and choose $k_o \in \mathbb{N}$ such that $\sum_{k=k_o}^{\infty} 2^{-k} < \varepsilon$. Define

$$y^l = \sum_{k=1}^l u_k y_k \quad (l = 1, 2, \dots).$$

Obviously any y^l has only finite number of coordinates different from zero. Let $l_o \geq k_o$ be such that $|u_k| \lambda \leq 1$ for any $k \geq l_o$. Such a number l_o exists because $u = (u_k) \in c_o$. We have for any $l \geq l_o$

$$\begin{aligned} I_\Phi(\lambda(P_1 u - y^l)) &= I_\Phi\left(\sum_{k=l+1}^{\infty} \lambda u_k y_k\right) \leq I_\Phi\left(\sum_{k=l+1}^{\infty} y_k\right) \\ &= \sum_{k=l+1}^{\infty} I_\Phi(y_k) \leq \sum_{k=l+1}^{\infty} 2^{-k} < \varepsilon, \end{aligned}$$

i.e. condition (11) holds. This means that $P_1 u \in h^\Phi$ for any $u \in c_o$. Applying (9), we get for any $u \in c_o$,

$$\begin{aligned} I_\Phi(P_1 u / \|u\|_\infty) &= \sum_{k=1}^{\infty} I_\Phi(u_k y_k / \|u\|_\infty) \\ &\leq \sum_{k=1}^{\infty} I_\Phi(y_k) \leq \sum_{k=1}^{\infty} 2^{-k} = 1, \end{aligned}$$

whence it follows that

$$(12) \quad \|P_1 u\|_\Phi \leq \|u\|_\infty \quad (\forall u \in c_o).$$

Let $k_o \in N$ be such that $|u_{k_o}| = \|u\|_\infty$. Then, in view of (10), we get

$$\begin{aligned} I_\Phi((1 + \varepsilon)\|u\|_\infty^{-1} P_1 u) &\geq I_\Phi((1 + \varepsilon)\|u\|_\infty^{-1} u_{k_o} y_{k_o}) \\ &= I_\Phi((1 + \varepsilon)y_{k_o}) > 1, \end{aligned}$$

whence it follows that

$$\|P_1 u\|_\Phi \geq \frac{1}{1 + \varepsilon} \|u\|_\infty \quad (\forall u \in c_o).$$

Defining now

$$Pu = (1 + \varepsilon)P_1 u \quad (\forall u \in c_o),$$

we get a linear operator from c_o into $h^\Phi(X)$ satisfying

$$\|u\|_\infty \leq \|Pu\|_\Phi \leq (1 + \varepsilon)\|u\|_\infty \quad (\forall u \in c_o),$$

which means that P is a $(1 + \varepsilon)$ -isometry. Since $\varepsilon > 0$ is arbitrary this means that c_o is embedded almost isometrically into h^Φ and the proof of statement (i) is finished.

(ii). Assume that $\Phi \in \delta_2^o$ and $X = \mathbb{R}$. Then $h^\Phi = l^\Phi$ and l^Φ is the dual space of h^{Φ^*} , where Φ^* is the Orlicz function complementary in the sense of Young to Φ (cf. [HY]). Assume that h^Φ contains an isomorphic copy of c_o . Then it contains (as a dual space) a copy of l^∞ (cf. [BP]). But this contradicts the fact that the norm $\|\cdot\|_\Phi$ is order continuous in h^Φ . This contradiction finishes the proof of statement (ii) and so of Theorem 3 as well. \square

Recall that a Banach lattice E is said to be a KB-space whenever every increasing bounded in the norm sequence of nonnegative elements in E is norm convergent to an element of E (cf. [AB] and [KA]).

Remark. It is known (cf. [AB, p. 227]) that if E is a Banach lattice that is not KB-space, then c_o is embeddable in E and conversely. The space h^Φ is a Banach lattice that is KB-space if and only if $\Phi \in \delta_2^o$. Therefore, h^Φ contains an isomorphic copy of c_o if and only if $\Phi \notin \delta_2^o$. It is worth to notice that if X is an arbitrary Banach space, then $h^\Phi(X)$ need not be a Banach lattice. However, in one direction an analogous result (cf. Theorem 3) still holds true.

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(Received February 16, 1993)