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Rectangular covers of products missing diagonals

YUKINOBU YAJIMA

Abstract. We give a characterization of a paracompact Σ-space to have a $G_δ$-diagonal in terms of three rectangular covers of $X^2 \setminus \Delta$. Moreover, we show that a local property and a global property of a space $X$ are given by the orthocompactness of $(X \times \beta X) \setminus \Delta$.

Keywords: Σ-space, $G_δ$-diagonal, σ-closure-preserving, σ-cushioned, rectangular cover, orthocompact, metacompact, Fréchet space

Classification: 54B10, 54D20, 54E18

1. Main theorem

All spaces in this paper are assumed to be regular $T_1$. The diagonal of a space $X$ is denoted by $\Delta$, that is, $\Delta = \{(x, x) : x \in X\}$.

Let $X$ be a space and $\mathcal{V}$ a collection of subsets of the square $X^2$. We say that $\mathcal{V}$ is rectangular if each member of $\mathcal{V}$ is a subset of the form $U \times W$ in $X^2$. Note that if $\mathcal{V}$ is a rectangular open cover of $X^2 \setminus \Delta$, then it covers $X^2 \setminus \Delta$ and each member of $\mathcal{V}$ is a subset of the form $U \times W$ such that $U$ and $W$ are disjoint open sets in $X$.

Gruenhage and Pelant [4] proved that a paracompact Σ-space $X$ has a $G_δ$-diagonal (i.e. is a σ-space), if $X^2 \setminus \Delta$ is paracompact. Subsequently, Kombarov [7] proved that a paracompact Σ-space $X$ has a $G_δ$-diagonal if and only if there is a locally finite rectangular open cover of $X^2 \setminus \Delta$.

Our main theorem is an extension of these results in terms of three rectangular covers of $X^2 \setminus \Delta$.

**Theorem 1.** The following are equivalent for a paracompact Σ-space $X$.

(a) $X$ has a $G_δ$-diagonal.
(b) There is a σ-locally finite rectangular open cover of $X^2 \setminus \Delta$.
(c) There is a σ-closure-preserving rectangular open cover of $X^2 \setminus \Delta$.
(d) There is a rectangular open cover of $X^2 \setminus \Delta$ which has a σ-cushioned open refinement.

The author announced Theorem 1 except (c) $\Rightarrow$ (a) in [10], and asked whether (c) implies (a) in the conference. Answering this, we give a complete proof of Theorem 1 in the next section.
2. Proof of Theorem 1

The main idea of the proof of Theorem 1 is based on Gruenhage-Pelant’s. In fact, it will be proceeded along the line of that of [4, Theorem 4]. Here, we have to do two kinds of parallel arguments.

Recall that a collection of $\mathcal{V}$ of subsets of a space $X$ is closure-preserving if

$$\bigcup \{ V : V \in \mathcal{V} \} = \bigcup \{ \overline{V} : V \in \mathcal{V}' \}$$

for each $\mathcal{V}' \subset \mathcal{V}$. We say that $\mathcal{V}$ is $\sigma$-closure-preserving if it can be written as $\mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n$ such that each $\mathcal{V}_n$ is closure-preserving.

**Lemma 1.** Let $X$ be a space with $p \in X$. Let $\mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n$ be a $\sigma$-closure-preserving rectangular open cover of $X^2 \setminus \Delta$. If there is a countable subset $M$ of $X \setminus \{ p \}$ such that $p \notin M$, then $p$ is a $G_{\delta}$-point.

**Proof:** Let $M = \{ x_n : n \in \omega \}$. Let $F_n = \bigcup \{ \overline{V} : V \in \mathcal{V}_n, j \leq n \text{ with } (p, x_n) \not\in \overline{V} \}$ for each $n \in \omega$. Then each $F_n$ is a closed subset in $X^2 \setminus \Delta$ such that $(p, x_n) \not\in F_n$. For each $n \in \omega$, take a basic open neighborhood $G_n \times H_n$ of $(p, x_n)$ in $X^2 \setminus \Delta$, disjoint from $F_n$. We show $\bigcap_{n \in \omega} G_n = \{ p \}$. Assume that there is some $y \in \bigcap_{n \in \omega} G_n$ with $y \neq p$. Take some $V = U \times W \in \mathcal{V}$ such that $(y, p) \in V$.

Choose $m \in \omega$ with $V \subseteq \mathcal{V}_m$. By $p \notin \mathcal{V}$, we have $(p, x_n) \notin \overline{V}$. Hence it follows that $\overline{V} \subseteq F_n$ for each $n \geq m$. Choose some $k \geq m$ with $x_k \in W$. Then it follows that $(y, x_k) \in U \times W = V \subseteq \bigcup_{n \in \omega} F_n$. On the other hand, by $(y, x_k) \in G_k \times H_k$, we have $(y, x_k) \notin F_k$. This is a contradiction. $\square$

Let $\mathcal{V}$ and $\mathcal{O}$ be two collections of subsets of a space $X$. Recall that $\mathcal{V}$ is cushioned in $\mathcal{O}$ if for each $V \in \mathcal{V}$, one can assign an $O(V) \in \mathcal{O}$ such that for each $V' \subset \mathcal{V}$, $\bigcup \{ V : V \in \mathcal{V}' \} \subseteq \bigcup \{ O(V) : V \in \mathcal{V}' \}$. Such an assignment $V \mapsto O(V)$, $V \in \mathcal{V}$, is called a cushioned assignment from $\mathcal{V}$ into $\mathcal{O}$. We say that $\mathcal{V}$ is $\sigma$-cushioned in $\mathcal{O}$ if it can be written as $\mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n$ such that each $\mathcal{V}_n$ is cushioned in $\mathcal{O}$.

**Lemma 2.** Let $X$ be a space with $p \in X$. Let $\mathcal{O}$ be a rectangular open cover of $X^2 \setminus \Delta$. Let $\mathcal{V}$ be a collection of open sets in $X^2 \setminus \Delta$ which is cushioned in $\mathcal{O}$. If there is a countable subset $M$ of $X \setminus \{ p \}$ such that $p \notin M$ and $M \times \{ p \} \subseteq \bigcup \mathcal{V}$, then $p$ is a $G_{\delta}$-point.

**Proof:** Let $V \mapsto O(V)$ be a cushioned assignment from $\mathcal{V}$ into $\mathcal{O}$, and let $M = \{ x_n : n \in \omega \}$. For each $n \in \omega$, take $V_n \in \mathcal{V}$ with $(x_n, p) \in V_n$, and let $W_n = \{ x \in X : (x, x_n) \in V_n \}$. It suffices to show $\bigcap_{n \in \omega} W_n = \{ p \}$. Assume that there is some $y \in \bigcap_{n \in \omega} W_n$ with $y \neq p$. Let $O(V_n) = P_n \times Q_n$ for each $n \in \omega$. By $(x_n, p) \in V_n \subseteq O(V_n)$, we have $p \notin \overline{P_n}$ for each $n \in \omega$. So it follows that $(p, y) \notin \bigcup_{n \in \omega} (P_n \times Q_n) = \bigcup_{n \in \omega} O(V_n) \supseteq \bigcup_{n \in \omega} \overline{V_n}$. There is an open neighborhood $G$ of $p$ such that $(G \times \{ y \}) \cap \bigcup_{n \in \omega} V_n = \emptyset$. By $(x_n, y) \in V_n$, each $x_n$ is not in $G$. Hence we have $p \notin \overline{M}$, which is a contradiction. $\square$

A space $X$ is called a $\Sigma$-space if there are a closed cover $\mathcal{C}$ of $X$ by countably compact sets, and a $\sigma$-discrete closed cover $\mathcal{F}$ of $X$ such that whenever $C \in \mathcal{C}$ and $U$ is open in $X$ with $C \subseteq U$, then $C \subseteq F \subseteq U$ for some $F \in \mathcal{F}$. 

Y. Yajima
The class of paracompact $\Sigma$-spaces is a broad one which is countably productive (see [3], [8]).

**Lemma 3.** Let $X$ be a $\Sigma$-space. If there is a $\sigma$-closure-preserving rectangular open cover $\mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n$ of $X^2 \setminus \Delta$, then each point of $X$ is $G_\delta$.

**Proof:** Let $\mathcal{C}$ and $\mathcal{F}$ be two closed covers of $X$, described as above. Assume that some $p \in X$ is not a $G_\delta$-point. As stated in the proof of [4, Lemma 3], there is a closed $G_\delta$-set $Y$ in $X$ containing $p$ such that $y \in Y$ and $y \in C \in \mathcal{C}$ implies $p \in C$. Then note that $p$ is not $G_\delta$ in $Y$. Let $V = U_Y \times W_Y$ for each $V \in \mathcal{V}$. Let $U_n = \{U_V : V \in \mathcal{V}_n \text{ with } p \in W_V\}$ for each $n \in \omega$. Note that each $U_n$ is closure-preserving in $X \setminus \{p\}$. Let $\mathcal{U} = \bigcup_{n \in \omega} U_n$. Then $\mathcal{U}$ is a $\sigma$-closure-preserving open cover of $X \setminus \{p\}$.

Now, we construct $\{U_\alpha, G_\alpha, z_\alpha : \alpha \in \omega_1\}$, satisfying the following conditions; for each $\alpha \in \omega_1$,

(i) $U_\alpha \in \mathcal{U}$ and $G_\alpha$ is an open set in $Y$,

(ii) $\overline{U_\alpha} \cap Y \subset G_\alpha \subset \overline{G_\alpha} \subset Y \setminus \{p\}$,

(iii) $z_\alpha \in (U_\alpha \cap (Y \setminus \{p\})) \setminus \bigcup_{\beta < \alpha} G_\beta$.

In fact, for $\alpha \in \omega_1$, assume that $\{U_\beta, G_\beta, z_\beta : \beta < \alpha\}$ satisfies the above conditions. Since $V \setminus \{p\}$ is not $F_\sigma$ in $Y$, $\{G_\beta \setminus \{p\} \} \beta < \alpha$ does not cover $Y \setminus \{p\}$. However, as $\mathcal{U}$ covers $X \setminus \{p\}$, we can choose the desired $z_\alpha$ and $U_\alpha$. By the choice of $U_n$, note that $p \notin \overline{U_\alpha}$. Since $X$ is regular, we can choose the desired $G_\alpha$.

Here we may assume without loss of generality that $\{U_\alpha : \alpha \in \omega_1\} \subset \mathcal{U}_m$ for some $m \in \omega$. Let $Z = \{z_\alpha : \alpha \in \omega_1\}$. Then $Z$ is uncountable. Moreover, $Z$ is closed discrete in $Y \setminus \{p\}$. For, pick any $x \in Z \setminus \{p\}$. Since $U_m$ is closure-preserving, $Z \setminus \{p\} \subset \bigcup_{\alpha \in \omega_1} \overline{U_\alpha}$. Let $\alpha_0 = \min\{\alpha \in \omega_1 : x \in \overline{U_\alpha}\}$. Let $N = G_{\alpha_0} \setminus \bigcup_{\beta < \alpha_0} G_\beta$. Then $N$ is an open neighborhood of $x$ in $Y$ such that $N \cap Z \subset \{z_\alpha\}$.

It follows from Lemma 1 that $p \notin \overline{N}$ for each countable subset $M$ of $Z$. The remaining argument is the same as in the proof of [4, Lemma 3].

**Lemma 4.** Let $X$ be a $\Sigma$-space. If there is a rectangular open cover $\mathcal{O}$ of $X^2 \setminus \Delta$ which has a $\sigma$-cushioned open refinement $\mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n$, then each point of $X$ is $G_\delta$.

**Proof:** Let $\mathcal{C}$, $\mathcal{F}$, $p$ and $Y$ be the same as in the above proof. Let $V \mapsto O(V)$, $V \in \mathcal{V}_n$, be a cushioned assignment of $\mathcal{V}_n$ into $\mathcal{O}$. Let $U_V = \{x \in X : (x, p) \in V\}$ and $O(V) = P_V \times Q_V$ for each $V \in \mathcal{V}$. Moreover, let $U_n = \{U_V : V \in \mathcal{V}_n\}$ and $\mathcal{P}_n = \{P_V : V \in \mathcal{V}_n \text{ with } p \in Q_V\}$ for each $n \in \omega$. Let $\mathcal{U} = \bigcup_{n \in \omega} U_n$ and $\mathcal{P} = \bigcup_{n \in \omega} \mathcal{P}_n$. Then $\mathcal{U}$ is an open cover of $X \setminus \{p\}$ and each $U_n$ is cushioned in $\mathcal{P}_n$ with the cushioned assignment $U_V \mapsto P_V$ in $X \setminus \{p\}$. Since $p \notin \overline{\mathcal{P}}$ for each $P \in \mathcal{P}$ and $Y \setminus \{p\}$ is not $F_\sigma$ in $Y$, $\{P \cap Y : P \in \mathcal{P}\}$ has no countable subcover of $Y \setminus \{p\}$. So we can inductively choose $\{U_\alpha, P_\alpha, z_\alpha : \alpha \in \omega_1\}$, satisfying for each $\alpha \in \omega_1$,

(iv) $U_\alpha \in \mathcal{U}$ and $P_\alpha \in \mathcal{P}$,

(v) $z_\alpha \in (U_\alpha \cap (Y \setminus \{p\})) \setminus \bigcup_{\beta < \alpha} P_\beta$. 

Rectangular covers of products missing diagonals 149
We may assume that \( \{ U_\alpha : \alpha \in \omega_1 \} \subset U_m \) and \( \{ P_\alpha : \alpha \in \omega_1 \} \subset P_m \) for some \( m \in \omega \). Then \( U_\alpha \mapsto P_\alpha, \alpha \in \omega_1 \), is a cushioned assignment. Let \( Z = \{ z_\alpha : \alpha \in \omega_1 \} \). Similarly, \( Z \) is closed discrete in \( Y \setminus \{ p \} \). Here, using Lemma 2 instead of Lemma 1, the remaining argument is the same as above.

A space \( X \) is called a \( \beta \)-space if there is a function \( g : \omega \times X \to \tau(X) \), where \( \tau(X) \) denotes the topology of \( X \), satisfying for each \( x \in X \),

(i) \( x \in \bigcap_{n \in \omega} g(n, x) \),
(ii) if \( x \in g(n, x_n) \) for each \( n \in \omega \), then \( \{ x_n : n \in \omega \} \) has a cluster point in \( X \).

Since \( \Sigma \)-spaces and semi-stratifiable spaces are \( \beta \)-spaces (see [3, Theorem 7.8 (i)]), the class of \( \beta \)-spaces is fairly broad.

The definition of \( W_\delta \)-diagonal in terms of the notion of a sieve is seen in [3], [4]. We do not restate it here. It is shown in [2] (or [3, Theorem 6.6]) that a submetacompact space with a \( W_\delta \)-diagonal has a \( G_\delta \)-diagonal.

**Lemma 5.** Let \( X \) be a \( \beta \)-space such that each point of \( X \) is \( G_\delta \). If there is a \( \sigma \)-closure-preserving rectangular open cover \( \mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n \) of \( X^2 \setminus \Delta \), then \( X \) has a \( W_\delta \)-diagonal.

**Proof:** Let \( g : \omega \times X \to \tau(X) \) be a function, described as above. Let \( h : \omega \times X \to \tau(X) \) be a function such that \( \bigcap_{n \in \omega} h(n, x) = \{ x \} \) for each \( x \in X \). As in the proof of [4, Theorem 4], we can construct a sieve \( (G, X^{<\omega}) \), satisfying the following: If \( s = \langle x_0, \ldots, x_{n-1} \rangle \in X^{<\omega} \) and \( x \in G(s) \), then \( G(s \setminus \langle x \rangle) \) is an open neighborhood of \( x \) such that

(i) \( G(s \setminus \langle x \rangle) \subset G(s) \cap g(n, x) \cap h(n, x) \),
(ii) if \( i < n \) and \( x_i \neq x \), then

\[
(\{ x_i \} \times G(s \setminus \langle x \rangle)) \cap \left( \bigcup \{ V : V \in \mathcal{V}_j, j < n \text{ with } (x_i, x) \notin \overline{V} \} \right) = \emptyset.
\]

Assume that \( \bigcap_{n \in \omega} G(s \upharpoonright n) \) contains two distinct points for some \( s = \langle x_0, x_1, \ldots \rangle \in X^{\omega} \). Then, by (i), no point of \( X \) is repeated infinitely many times in the sequence \( s \). By the choice of \( g \) and (i), \( \{ x_n : n \in \omega \} \) has a cluster point \( y \). Then we have \( y \in \bigcap_{n \in \omega} G(s \upharpoonright n) = \bigcap_{n \in \omega} G(s \upharpoonright n) \). There is some \( z \in \bigcap_{n \in \omega} G(s \upharpoonright n) \) with \( y \neq z \). Choose an \( n_0 \in \omega \) and a \( V_0 = U \times W \in \mathcal{V}_{n_0} \) with \( (y, z) \in V_0 \). Find some \( k, m \in \omega \) such that \( m > k > n_0 \), \( x_k \neq x_m \) and \( \{ x_k, x_m \} \subset U \). By \( x_m \notin \overline{W} \), note \( (x_k, x_m) \notin \overline{V_0} \). By (ii), we have \( (\{ x_k \} \times G(s \upharpoonright m + 1)) \cap \overline{V_0} = \emptyset \). On the other hand, we have \( (x_k, z) \in (\{ x_k \} \times G(s \upharpoonright m + 1)) \). This is a contradiction.

**Lemma 6.** Let \( X \) be a \( \beta \)-space such that each point of \( X \) is \( G_\delta \). If there is a rectangular open cover \( \mathcal{O} \) of \( X^2 \setminus \Delta \) which has a \( \sigma \)-cushioned open refinement \( \mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n \), then \( X \) has a \( W_\delta \)-diagonal.

**Proof:** Let \( V \mapsto O(V) \) be a cushioned assignment from \( \mathcal{V}_n \) into \( \mathcal{O} \) for each \( n \in \omega \). Let \( g \) and \( h \) be the same functions as above. Moreover, we can also construct
a similar sieve \((G, X^<\omega)\) as above, where we only replace the condition (ii) with the following;

(ii') if \(i < n\) and \(x_i \neq x\), then

\[
(\{x_i\} \times G(s \cap (x))) \cap \left( \bigcup \{V \in \mathcal{V}_j, j \leq n, (x_i, x) \notin O(V)\} \right) = \emptyset.
\]

Take \(s, y\) and \(z\) as above. Choose an \(n_0 \in \omega\) and a \(V_0 \in \mathcal{V}_{n_0}\) with \((y, z) \in V_0\). Take an open neighborhood \(U\) of \(y\) such that \(U \times \{z\} \subset V_0\). Find some \(k, m \in \omega\) with \(m > k > n\), \(x_k \neq x_m\) and \(\{x_k, x_m\} \subset U\). Let \(O(V_0) = P \times Q\). Since \(x_m \in U \subset P\), it follows that \((x_k, x_m) \notin O(V_0)\). By (ii'), we have \((\{x_k\} \times G(s \upharpoonright m + 1)) \cap V_0 = \emptyset\).

This is a contradiction. \(\square\)

We say that an open cover \(O\) of \(X^2 \setminus \Delta\) is \textit{rectangular cozero} if each member of \(O\) is a subset of the form \(P \times Q\) such that \(P\) and \(Q\) are disjoint cozero sets in \(X\).

Since each open \(F_\sigma\)-set in a normal space is exactly a cozero set, so is each open set in a metric space. So, Kombarov [7] actually showed the following.

**Lemma 7.** If a paracompact space \(X\) has a \(G_\delta\)-diagonal, then there is a locally finite rectangular cozero cover of \(X^2 \setminus \Delta\).

Now, we complete the proof of our main theorem.

**Proof of Theorem 1:** (a) \(\Rightarrow\) (b): This follows from Lemma 7 (or [7, Theorem 1]).

(b) \(\Rightarrow\) (c): Obvious.

(a) \(\Rightarrow\) (d): Since a \(\sigma\)-locally finite rectangular cozero cover of \(X^2 \setminus \Delta\) has a \(\sigma\)-cushioned (rectangular) open refinement, this also follows from Lemma 7.

(c) \(\Rightarrow\) (a): Remember that each \(\Sigma\)-space is a \(\beta\)-space, and that a submeta-
compact space has a \(G_\delta\)-diagonal iff it has a \(W_\delta\)-diagonal. So this follows from
Lemmas 3 and 5.

(d) \(\Rightarrow\) (a): Similarly, this follows from Lemmas 4 and 6. \(\square\)

3. Orthocompactness of \((X \times \beta X) \setminus \Delta\)

Arhangel’skiï and Kombarov [1] proved that a compact space \(X\) is first count-
able if \(X^2 \setminus \Delta\) is normal. First, we consider what local property of a compact space \(X\) can be obtained if the normality of \(X^2 \setminus \Delta\) is replaced by the orthocompactness of it. For this, we also use some rectangular open covers.

Recall that an open cover \(\mathcal{V}\) of a space \(X\) is \textit{interior-preserving} if \(\bigcap \mathcal{V}'\) is open in \(X\) for each \(\mathcal{V}' \subset \mathcal{V}\).

A space \(X\) is called a \textit{Fréchet space} if for each \(p \in X\) and each subset \(M\) of \(X\) with \(p \in \overline{M}\), there is a sequence \(\{x_n\}\) of points in \(M\) which converges to \(p\).
Note that point-finite open covers of a space are interior-preserving, and that first countable spaces and Lašnev spaces are Fréchet.

For a collection $V$ of open sets in a product $X \times C$ and an $(x, y) \in X \times C$, let $\bigcap V(x, y) = \bigcap \{V \in V : (x, y) \in V\}$.

**Theorem 2.** Let $C$ be a countably compact space and $X$ a subspace of $C$. If there is a rectangular open cover of $\left( X \times C \right) \setminus \Delta$ which has an interior-preserving open refinement, then $X$ is a Fréchet space.

**Proof:** Let $M$ be a subset of $X$ with $p \in \overline{M} \setminus M$. Let $O$ be a rectangular open cover of $\left( X \times C \right) \setminus \Delta$ and $\mathcal{V}$ an interior-preserving open refinement of $O$. Since $p$ is not isolated in $X$ and each $x \in \bigcap \mathcal{V}(p, x)$ is an open neighborhood of $(p, x)$ in $X \times C$, we can inductively choose a sequence $\{x_n : n \in \omega\}$ of distinct points in $M$ such that $(x_n, x_i) \in \bigcap \mathcal{V}(p, x_i)$ for each $i < n$ and each $n \in \omega$. We show that $\{x_n : n \in \omega\}$ converges to $p$. Assume the contrary. This is an open neighborhood $U$ of $p$ in $X$ such that $x_n \not\in U$ for infinitely many $n$’s. There is a cluster point $y$ of $\{x_n \in X \setminus U : n \in \omega\}$ in $C$. By $y \neq p$, we can find a $V \in \mathcal{V}$ and an $O = P \times Q \in \mathcal{O}$ such that $(p, y) \in V \subset O$. Take an open neighborhood $W$ of $y$ in $C$ such that $\{p\} \times W \subset \bigcap \mathcal{V}(p, y)$. Moreover, take some $k, m \in \omega$ such that $k < m$ and $\{x_k, x_m\} \subset W$. Since $(p, x_k) \in \bigcap \mathcal{V}(p, y)$, it follows that $\bigcap \mathcal{V}(p, x_k) \subset \bigcap \mathcal{V}(p, y)$. Hence we have $x_m \in P \cap Q$. This is a contradiction. □

The author first showed in Theorem 2 that $X$ has countable tightness. Subsequently, N. Kemoto kindly pointed out that $X$ is a Fréchet space.

We say that a space $X$ is *orthocompact* if every open cover of $X$ has an interior-preserving open refinement.

As an analogue of [1, Theorem 10], we immediately have

**Corollary 1.** Let $X$ be a countably compact space. If $X^2 \setminus \Delta$ is orthocompact, then $X$ is a Fréchet space.

For a Tychonoff space $X$, we denote by $\beta X$ the Stone-Čech compactification of $X$. Junnila [5] proved that the orthocompactness of $X \times \beta X$ gives the metacompactness of $X$. Finally, we show that the orthocompactness of $(X \times \beta X) \setminus \Delta$ gives not only the local property of $X$ but also the global property of $X$.

**Theorem 3.** Let $X$ be a Tychonoff space and $\gamma X$ a compactification of $X$. If $(X \times \gamma X) \setminus \Delta$ is orthocompact, then $X$ is metacompact.

**Proof:** The proof is obtained by modifying that of [9, Theorem 2.2]. Let $U, U^*$, $V$ and $G$ be the same ones as in the proof of it. There is an interior-preserving
open refinement $\mathcal{H}$ of $\mathcal{G}(X \times \gamma X) \setminus \Delta$. For each $x \in X$, fix a $V_x \in \mathcal{V}$ with $x \in V_x$. For each $(x, x') \in (X \times \gamma X) \setminus \Delta$, we take a basic open neighborhood $P_{x,x'} \times Q_{x,x'}$ of $(x, x')$ which is contained in some member of $\mathcal{H}$. Pick $x \in X$. Since $\gamma X \setminus V_x$ is compact, there is a finite subset $F(x)$ of $\gamma X \setminus V_x$ such that $\gamma X \setminus V_x \subseteq \bigcup_{z \in F(x)} Q_{x,z}$. Let $W_x = (\bigcap_{z \in F(x)} P_{x,z}) \cap V_x$. Here, we set $\mathcal{W} = \{W_x : x \in X\}$. It suffices from [6, Theorem 3.6] to show that there is a finite subcollection $\mathcal{U}_x$ of $\mathcal{U}$ such that $\text{St}(x, \mathcal{W}) \subseteq \bigcup \mathcal{U}_x$ and $x \in \bigcap \mathcal{U}_x$ for each $x \in X$. For this, it also suffices to show that $\text{Cl}_x \text{St}(x, \mathcal{W}) \subseteq \text{St}(x, \mathcal{U}^*)$ for each $x \in X$. Assuming the contrary, we pick some $x \in X$ and some $q \in \text{Cl}_x \text{St}(x, \mathcal{W}) \setminus \text{St}(x, \mathcal{U}^*)$. By $x \neq q$, $\bigcap \mathcal{H}(x, q)$ is an open neighborhood of $(x, q)$. Take a basic open neighborhood $S \times T$ of $(x, q)$ contained in $\bigcap \mathcal{H}(x, q)$. Pick $p \in T \cap \text{St}(x, \mathcal{W})$, and pick $y \in X$ with $x \in W_y$ and $p \in W_y$. Since $x \in W_y \subseteq V_y \subseteq U_{V_y}^* \subseteq \mathcal{U}^*$, it follows that

$$q \in \gamma X \setminus \text{St}(x, \mathcal{U}^*) \subseteq \gamma X \setminus V_y \subseteq \bigcup_{z \in F(y)} Q_{y,z}.$$

Find $z \in F(y)$ with $q \in Q_{y,z}$. By the same argument as in the proof of [9, Theorem 2.2], we obtain that $\{(x, q), (x, p), (p, q)\} \subseteq H_0$ for some $H_0 \in \mathcal{H}$, and so that $\{(x, p), (p, q)\} \subseteq (V_0 \cap X) \times (\gamma X \setminus \text{Cl}_x \gamma X V_0)$ for some $V_0 \in \mathcal{V}$. This is a contradiction. \hfill \Box

By Theorems 2 and 3, we obtain

**Corollary 2.** Let $X$ be a Tychonoff space. If $(X \times \beta X) \setminus \Delta$ is orthocompact, then $X$ is metacompact and Fréchet.

**References**


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