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Ideals in selfdistributive groupoids

Tomáš Kepka

Abstract. Products of (left) ideals in selfdistributive groupoids are studied.

Keywords: groupoid, distributive, ideal

Classification: 20N02

The purpose of this very short note is to complete some results from [1]. Other results on, comments about and aspects of left distributive groupoids (and further references as well) may be found in [2], [4] and [5].

1. Introduction

1.1. A groupoid is a non-empty set supplied with a binary operation.

Let $G$ be a groupoid and let $\mathcal{P}(G)$ denote the set of all subsets of $G$. Then we define a binary operation on $\mathcal{P}(G)$ by $AB = \{ab; a \in A, b \in B\}$ for all $A, B \in \mathcal{P}(G)$. In this way, $\mathcal{P}(G)$ becomes a groupoid and we denote by $\mathcal{R}(G)$ the subgroupoid of $\mathcal{P}(G)$ generated by $G$. Clearly, $\mathcal{R}(G)$ is trivial iff $G = G^2$.

A non-empty subset $I$ of $G$ is said to be a left (right) ideal of $G$ if $GI \subseteq I$ ($IG \subseteq I$). We denote by $\mathcal{I}_l(G)$ ($\mathcal{I}_r(G)$) the set of left (right) ideals of $G$.

A non-empty subset $I$ of $G$ is said to be an ideal if it is both a left and right ideal of $G$. We denote by $\mathcal{I}(G)$ the set of ideals of $G$.

1.2. Let $G$ be a groupoid. We put $G^{(1)} = G$ and $G^{(n+1)} = G \cdot G^{(n)}$ for every $n \geq 1$. Further, $\mathcal{Q}(G) = \{G^{(n)}; n \geq 1\} \subseteq \mathcal{R}(G)$.

Similarly, let $G^{(n,0)} = G^{(n)}$ and $G^{(n,m+1)} = G^{(n,m)} \cdot G$ for every $n \geq 1$ and every $m \geq 0$.

1.3. A groupoid $G$ is said to be

- left distributive if $a \cdot bc = ab \cdot ac$ for all $a, b, c \in G$;
- right distributive if $bc \cdot a = ba \cdot ca$ for all $a, b, c \in G$;
- distributive if it is both left and right distributive;
- medial if $ab \cdot cd = ac \cdot bd$ for all $a, b, c, d \in G$. 
2. Examples

2.1 Example. Let $D_0$ designate the set of ordered pairs $(n,m)$, where $n, m$ are integers, $n \geq 1$, $n \neq 2$ and $m \geq 0$. Now define a multiplication on $D_0$ as follows: $(n,m)(k,l) = (3,0)$ if $l \geq 1$; $(n,m)(k,0) = (k+1,0)$ if $k \geq 3$; $(n,m)(1,0) = (n,m+1)$. Then $D_0$ becomes a groupoid and it is easy to check that $D_0$ is a left distributive groupoid. Moreover, $D_0$ is medial, $D_0$ does not contain any idempotent element and $uvz \neq zuv$ for all $u, v, z \in D_0$; in particular, $D_0$ is not right distributive. Further, notice that $D_0$ is generated by the element $(1,0)$. Finally, define a relation $\leq_0$ on $D_0$ by $(n,m) \leq_0 (k,l)$ iff at least one of the following cases takes place: $k \leq n$, $m = l$; $3 \leq n$, $0 \leq m < l$; $3 \leq n$, $k = 1$; $k = 1$, $0 \leq l < m$. Then $\leq_0$ is a linear ordering of $D_0$ and this ordering is stable with respect to the operation of the groupoid $D_0$.

2.2 Example. Consider the following three-element groupoid $G$:

\[
\begin{array}{c|ccc}
  & 0 & 1 & 2 \\
 0 & 1 & 2 & 2 \\
 1 & 1 & 2 & 2 \\
 2 & 1 & 2 & 2 \\
\end{array}
\]

Then $G$ is left distributive, $\mathcal{R}(G) = \mathcal{I}_l(G) = \{G^{(1)}, G^{(2)}, G^{(3)}\}$ and $G^{(3)}$ is not a right ideal.

2.3 Example. Consider the following four-element groupoid $G$:

\[
\begin{array}{c|cccc}
  & 0 & 1 & 2 & 3 \\
 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 3 & 0 \\
 2 & 0 & 0 & 1 & 0 \\
 3 & 0 & 0 & 3 & 0 \\
\end{array}
\]

Then $G$ is left distributive, $\mathcal{R}(G) = \{G^{(1,0)}, G^{(1,1)}, G^{(1,2)}, G^{(3,0)}\} = \mathcal{I}(G) = \mathcal{I}_r(G) \neq \mathcal{I}_l(G) = \mathcal{R}(G) \cup \{A\}$, where $A = \{0, 1\}$ is a left ideal but not a right ideal; $\mathcal{I}_l(G)$ is not linearly ordered by inclusion.

2.4 Example. Consider the following three-element groupoid $G$:

\[
\begin{array}{c|ccc}
  & 0 & 1 & 2 \\
 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0 \\
 2 & 0 & 0 & 0 \\
\end{array}
\]

Then $G$ is distributive, $\mathcal{R}(G) = \{G^{(1)}, G^{(2)}\} \neq \mathcal{I}(G)$ and $\mathcal{I}(G)$ is not linearly ordered by inclusion.
2.5 Example. Consider the following three-element groupoid $G$:

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</table>

Then $G$ is left distributive and $G$ is both left and right-ideal-free. Moreover, $G$ is a left quasigroup but it is not a right quasigroup.

2.6 Example. Consider the following three-element groupoid $G$:

<table>
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<tr>
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</tbody>
</table>

Then $G$ is distributive and left-ideal-free. Moreover, $G$ is neither a left nor a right quasigroup.

2.7 Remark. By [3, 5.10], every finite left and right-ideal-free distributive groupoid is a quasigroup.

3. First observations on ideals of left distributive groupoids.

3.1 Lemma. Let $I$, $J$, $K$ be left ideals of a left distributive groupoid $G$. Then:

(i) $IJ$ is a left ideal and $IJ \subseteq J$.
(ii) $I \cdot JK = IJ \cdot IK$.
(iii) $I(J \cup K) = IJ \cup IK$ and $(J \cup K)I = JI \cup KI$.
(iv) If $J \subseteq K$, then $IJ \subseteq IK$ and $JI \subseteq KI$.

3.2 Lemma. Let $G$ be a left distributive groupoid such that $G = G^2$.

(i) If $I$ is a right ideal and $J$ is an ideal of $G$, then $IJ$ is a right ideal and $IJ \subseteq I \cap J$.
(ii) If $I$, $J$ are ideals of $G$, then $IJ$ is an ideal and $IJ \subseteq I \cap J$.

3.3 Proposition. Let $G$ be a left distributive groupoid. Then:

(i) The set $\mathcal{I}_l(G)$ of left ideals of $G$ is a subgroupoid of $\mathcal{P}(G)$ and $\mathcal{I}_l(G)$ is again a left distributive groupoid.
(ii) $\mathcal{R}(G)$ is a subgroupoid of $\mathcal{I}_l(G)$.
(iii) If $G = G^2$, then $\mathcal{I}(G)$ is a subgroupoid of $\mathcal{I}_l(G)$ and $\mathcal{I}(G)$ is a medial groupoid.
(iv) If $G$ is idempotent, then $\mathcal{I}_l(G)$ is idempotent and $\mathcal{I}(G)$ is a semilattice.
4. The groupoid $\mathcal{R}(G)$.

4.1 Lemma. Let $G$ be a left distributive groupoid and $A \in \mathcal{R}(G)$. Then:

(i) $GA \subseteq A$.

(ii) If $A \neq G$, then $G^{(n)} \cdot A = GA$ for every $n \geq 1$.

(iii) There exists $m \geq 1$ such that $G^{(m)} \subseteq A$.

Proof: (i) $A$ is a left ideal by 3.3 (ii).

(ii) Let $F$ be an absolutely free groupoid over a one-element set $\{x\}$ and let $f : F \to \mathcal{R}(G)$ be the uniquely determined homomorphism such that $f(x) = G$. Since $A \neq G$, we have $G \neq G^2$ and $A = f(r)$ for some $r \in F$, $l(r) \geq 2$; here, $l(r)$ means the length of $r$. Now, we shall proceed by induction on $l(r) + n$.

First, let $l(r) = 2$. Then $A = G^2$ and $G^{(3)} = G^{(n)} \cdot G^2 = (G^{(n)} G) (G^{(n)} G) = ((G^{(n)} G) G^{(n)} G) G^{(n)} G \subseteq G^{(n+1)} \cdot G^2$. The inclusion $G^{(n+1)} \cdot G^2 \subseteq G^{(3)}$ is evident, and hence $G^{(n+1)} \cdot G^2 = G^{(3)}$.

Next, let $r = sx$, $l(s) \geq 2$, $B = f(s)$. Then $GA = G^{(n)} \cdot BG = (G^{(n)} B) (G^{(n)} G) = ((G^{(n)} B) G^{(n)} G) G^{(n)} G \subseteq G^{(n+1)} \cdot BG = G^{(n+1)} \cdot A$. Then $GA = G^{(n+1)} \cdot A$. Similarly, if $r = xt$, $l(t) \geq 2$, $B = f(t)$. Then $G^{(n)} \cdot A = (G^{(n)} B) (G^{(n)} C) = GB \cdot GC = G \cdot BC = GA$.

(iii) We can assume that $A = BC$ and that $G^{(n)} \subseteq B \cap C$ for some $n \geq 2$. Then $G^{(n)} \cdot G^{(n)} \subseteq A$. However, by (ii), $G^{(n)} \cdot G^{(n)} = G^{(n+1)}$. $\square$

4.2 Lemma. Let $G$ be a left distributive groupoid. Then $G^{(n,m)} \cdot G^{(k)} = G^{(k+1)}$ for all $n \geq 1$, $m \geq 0$ and $k \geq 2$.

Proof: We can assume that $G \neq G^2$. Now, for $m = 0$, our equality follows from 4.1 (ii).

Let $k = 2$. We shall proceed by induction on $m$. We have $G^{(3)} = G^{(n,m)} \cdot G^2 = (G^{(n,m)} G) (G^{(n,m)} G) \subseteq G^{(n,m+1)} \cdot G^2 \subseteq G^{(3)}$, and so $G^{(3)} = G^{(n,m+1)} \cdot G^2$.

Let $k \geq 3$. Again, we shall proceed by induction on $m$. We have $G^{(k+1)} = G^{(n,m)} \cdot G^{(k)} = G^{(n,m)} \cdot (G \cdot G^{(k-1)}) = (G^{(n,m)} G) (G^{(n,m)} G^{(k-1)}) = G^{(n,m+1)} \cdot G^{(k)}$. $\square$

4.3 Lemma. Let $G$ be a left distributive groupoid. Then $G \cdot G^{(n,m)} = G^{(3)}$ for all $n \geq 1$, $m \geq 1$.

Proof: Assuming $G \neq G^2$, we shall proceed by induction on $m$. Now, $G \cdot G^{(n,m)} = (G \cdot G^{(n,m-1)}) \cdot G^2$. If $m \geq 2$, then $G \cdot G^{(n,m-1)} = G^{(3)}$ by induction and $G^{(3)} \cdot G^2 = G^{(3)}$ by 4.2. If $m = 1$, then $G \cdot G^{(n,1)} = G^{(n+1)}$ and our result follows from 4.2 again. $\square$

4.4 Lemma. Let $G$ be a left distributive groupoid. Then $G^{(n,m)} \cdot G^{(k,l)} = G^{(3)}$ for all $n \geq 1$, $m \geq 0$, $k \geq 1$, $l \geq 1$.

Proof: Using 4.1, 4.2 and 4.3, the result follows easily by induction on $l$. $\square$
4.5 Proposition ([1]). Let $G$ be a left distributive groupoid. Then:

(i) $G^{(n,m)} \cdot G^{(k,l)} = G^{(3)}$ for all $n \geq 1$, $m \geq 0$, $k \geq 1$, $l \geq 1$.
(ii) $G^{(n,m)} \cdot G^{(k,0)} = G^{(k+1,0)}$ for all $n \geq 1$, $m \geq 0$, $k \geq 2$.
(iii) $G^{(n,m)} \cdot G^{(1,0)} = G^{(n,m+1)}$ for all $n \geq 1$, $m \geq 0$.

Proof: See the preceding lemmas. □

4.6 Corollary. Let $G$ be a left distributive groupoid. Then:

(i) $\mathcal{R}(G) = \{G^{(n,m)}; n \geq 1, m \geq 0\}$.
(ii) If $G \neq G^2$, then $\mathcal{Q}(G) - \{G\} = \{G^{(k)}; k \geq 2\}$ is a left ideal of $\mathcal{R}(G)$.

4.7 Theorem. Let $G$ be a left distributive groupoid. Define a mapping $f : D_0 \rightarrow \mathcal{R}(G)$ by $f(n,m) = G^{(n,m)}$. Then

(i) $f$ is a projective homomorphism of the left distributive groupoids.
(ii) If $(n,m), (k,l) \in D_0$ and $(n,m) \leq_0 (k,l)$, then $G^{(n,m)} \subseteq G^{(k,l)}$.

Proof: (i) See 4.5 and 2.1. (ii) First, let $k \geq n$, $m = 1$. We have $G^{(n)} = (G \ldots (G \cdot G^{(k)}))$, where $G$ appears $(n-k)$-times, and hence $G^{(n)} \subseteq G^{(k)}$, since $G^{(k)}$ is a left ideal. This also implies that $G^{(n,m)} \subseteq G^{(k,l)}$.

Next, let $3 \leq n$ and $0 \leq m < l$. If $m = 0$, then $G^{(n,0)} \subseteq G^{(3)} = G \cdot G^{(k,l)} \subseteq G^{(k,l)}$. If $m \geq 1$, then $G^{(n,0)} \subseteq G^{(k,l-m)}$, and therefore $G^{(n,m)} = ((G^{(n,0)} \cdot G) \ldots G) \subseteq ((G^{(k,l-m)} \cdot G) \ldots G) = G^{(k,l)}$.

Finally, let $3 \leq n$ and $k = 1$. With respect to the preceding case, we can assume that $l \leq m$. Now, $G^{(n,m)} = ((G^{(n,m-l)} \cdot G) \ldots G) \subseteq ((GG) \ldots G) = G^{(1,l)}$. Similarly, if $k = 1$ and $0 \leq l < m$. □

4.8 Corollary. Let $G$ be a left distributive groupoid. Then $\mathcal{R}(G)$ is a medial left distributive groupoid which is linearly ordered by inclusion; this ordering is stable.

References


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