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Opial’s property and James’ quasi-reflexive spaces

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Abstract. Two of James’ three quasi-reflexive spaces, as well as the James Tree, have the uniform \( w^* \)-Opial property.

Keywords: fixed points, James’ quasi-reflexive spaces, James Tree, nonexpansive mappings, Opial’s property, the demiclosedness principle

Classification: Primary 46B20, 47H10

Introduction

Let \((X, \| \cdot \|)\) be a Banach space with a Schauder finite dimensional decomposition (FDD) [1], [20]. Define \( \beta_p((X, \| \cdot \|)) \) for \( p \in [1, \infty) \) to be the infimum of the set of numbers \( \lambda \) such that
\[
(\| x \|^p + \| y \|^p)^{1/p} \leq \lambda \| x + y \|
\]
for every \( x \) and \( y \) in \( X \) with \( \text{supp}(x) < \text{supp}(y) \) (we use here the notation in [1], [15]). In [15] M.A. Khamsi proved the following result.

**Theorem A.** Let \((X, \| \cdot \|)\) be a Banach space with a finite codimensional subspace \( Y \) such that \( \beta_p((Y, \| \cdot \|)) < 2^{1/p} \) for some \( p \in [0, \infty) \). Then \( X \) has weak normal structure.

He then used this theorem to deduce that James quasi-reflexive space, which consists of all null sequences \( x = \{x^i\} = \sum_{i=1}^\infty e_i \) (\( \{e_i\} \) is the standard basis in \( c_0 \)) for which the squared variation
\[
\sup_{p_1 < \cdots < p_m} \left[ \sum_{j=2}^m \left| \frac{\sum_{i=1}^\infty |x_i^{p_j} - x_{i+1}^{p_j-1}|^2}{\sum_{i=1}^\infty |x_i^{p_j}|^2} \right|^{1/2} \right]
\]
is finite, with the norm \( \| \cdot \| \) given by (2), has weak normal structure by claiming that \( \beta_2((J, \| \cdot \|_1)) = 1 \). As a matter of fact, \( \beta_2((J, \| \cdot \|_1)) \geq 2^{1/2} \), which can be easily seen by taking \( x = e_2 \) and \( y = e_3 \). Fortunately, Theorem A remains true with a slight modification of the definition of \( \beta_p((X, \| \cdot \|)) \), namely
\[
\tilde{\beta}((X, \| \cdot \|)) = \inf_{k=0,1,2,\ldots} \left\{ \inf \{ \lambda : (1) \text{ is valid for } x, y \in X \text{ with } \text{supp}(x) + k < \text{supp}(y) \} \right\},
\]
and \( \tilde{\beta}_2((J, \| \cdot \|)) = 1 \). Thus \((J, \| \cdot \|_1)\) does indeed possess weak normal structure. In the present paper we will prove that \((J, \| \cdot \|_1)\) has, in fact, the uniform \( w^* \)-Opial property [23], which, of course, also implies weak normal structure [1], [3], [4], [8].
Definitions and notations

We recall that a (dual) Banach space \((X, \| \cdot \|)\) has the \((w^*)\)-Opial property if whenever a sequence \(\{x_n\}\) in \(X\) converges weakly (weakly*) to \(x_0\), then for \(x \neq x_0\)
\[
\liminf_n \|x_0 - x_n\| < \liminf_n \|x - x_n\|
\]
[21], [22]. Opial’s property plays an important role in the study of weak convergence of iterates and random products of nonexpansive mappings and of the asymptotic behavior of nonlinear semigroups [4], [5], [7], [13], [18], [21], [22]. Moreover, it can be introduced in the open unit ball of a complex Hilbert space, equipped with the hyperbolic metric, where it is useful in proving the existence of fixed points of holomorphic self-mappings of \(B\) [5], [6].

The (dual) Banach space \((X, \| \cdot \|)\) is said to have the uniform \((w^*)\)-Opial property [23] if for every \(c > 0\) there exists an \(r > 0\) such that
\[
1 + r \leq \liminf_n \|x - x_n\|
\]
for each \(x \in X\) with \(\|x\| \geq c\) and every sequence \(\{x_n\}\) with \(w\-\lim_n x_n = 0\) \(\text{and } \liminf_n \|x_n\| \geq 1\).

In the linear space \(J\) defined by (2) one uses three different, but equivalent norms, \(\| \cdot \|_1\) (defined by (2)), \(\| \cdot \|_2\), and \(\| \cdot \|_3\), introduced by R.C. James [9], [10], [11]:
\[
\|x\|_2 = \sup_{k} \left(\sum_{j=1}^{k} |x^{p_{2j}} - x^{p_{2j-1}}|^2\right)^{1/2},
\]
\[
\|x\|_3 = \sup_{m} \left(\sum_{j=2}^{m} |x^{p_j} - x^{p_{j-1}}|^2 + |x^{p_m} - x^{p_1}|^2\right)^{1/2}.
\]
The choice of norms depends on one’s goals [2], [9], [10], [11], [20].

In [14] M.A. Khamsi used the ultraproduct method to prove that \((J, \| \cdot \|_3)\) has the fixed point property for nonexpansive mappings (FPP), i.e. for every nonempty weakly compact convex subset \(C\) of \((J, \| \cdot \|_3)\) any nonexpansive self-mapping \(T : C \to C\) has a fixed point. D. Tingley [24] has recently shown that \((J, \| \cdot \|_3)\) has, in fact, weak normal structure ([3]): every nonempty weakly compact convex subset \(C\) of \((J, \| \cdot \|_3)\) with \(\text{diam } C > 0\) has a nondiametral point \(y\), i.e.
\[
\sup_{x \in C} \|y - x\|_3 < \text{diam } C.
\]
This property immediately guarantees the FPP [17]. The proof of weak normal structure is based on the following property of weakly convergent sequences in \((J, \| \cdot \|_3)\): if \(\{x_n\}\) converges weakly to 0 and \(\text{diam } \{x_n\} > 0\), then
\[
\sup_m \left(\limsup_n \|x_m - x_n\|_3\right) > \liminf_n \|x_n\|_3.
\]
But it is easy to observe that the sequence \(-e_n + e_{n+1}\) tends weakly to 0 in 
\((J, \| \cdot \|_3)\) and
\[
\| \frac{1}{3}e_1 + e_n - e_{n+1} \|_3 = \| -e_n + e_{n+1} \|_3 = \sqrt{8}
\]
for \(n \geq 3\). Therefore \((J, \| \cdot \|_3)\) does not have Opial’s property.

**Main result**

In this section we are concerned with the spaces \((J, \| \cdot \|_1)\) and \((J, \| \cdot \|_2)\).

The predual Banach space \(I\) to \((J, \| \cdot \|_j), j = 1, 2,\) is generated by the biorthogonal functionals \(\{f_n\}\) to the basis \(\{u_n\} = \{e_1 + \cdots + e_n\}\) [12], [19]. Throughout this paper we will always treat \(J\) as \(I^*\).

**Theorem.** For \(j = 1, 2\) the space \((J, \| \cdot \|_j)\) has the uniform \(w^*-\)Opial property.

**Proof:** Let \(k \in \mathbb{N}\) and let \(P_k\) and \(Q_k\) be the natural projections in \(J\) associated with the basis \(\{u_n\}\):

\[
P_kx = \sum_{n=1}^{k} \xi^n u_n
\]

and

\[
Q_kx = \sum_{n=k+1}^{\infty} \xi^n u_n
\]

for each \(x = \sum_{n=1}^{\infty} \xi^n u_n \in J\). Note that if \(x = \sum_{n=1}^{\infty} \xi^n u_n\), then

\[
\|x\|_1 = \sup_{p_1 < \cdots < p_m} \left\{ \sum_{j=2}^{m} \left[ \sum_{n=p_{j-1}}^{p_j-1} \xi^n \right]^2 \right\}^{1/2}
\]

and

\[
\|x\|_2 = \sup_{p_1 < \cdots < p_{2k}} \left\{ \sum_{j=1}^{k} \left[ \sum_{n=p_{2j-1}}^{p_{2j}-1} \xi^n \right]^2 \right\}^{1/2}
\]

[11], [12]. Directly from these formulas we obtain

\[
\|x\|_j = \lim_k \|P_kx\|_j
\]

and

\[
\lim_k \|Q_kx\|_j = 0
\]
for all \( x \in J \) and \( j = 1, 2 \). Assume that a sequence \( \{x_n\} \) in \((J, \| \cdot \|_j)\) converges weakly* to 0 and let \( x \in J \). Then we have

\[
\lim_n \|P_k x_n\|_j = 0,
\]

\[
\liminf_n \|Q_k x_n\|_j = \liminf_n \|x_n\|_j,
\]

and

\[
\liminf_n \|x - x_n\| \geq \liminf_n \left[ \|P_k x - Q_{k+1} x_n\|_j - \|Q_k x\|_j - \|P_{k+1} x_n\|_j \right]
\]

\[
= \liminf_n \left[ \|P_k x - Q_{k+1} x_n\|_j - \|Q_k x\|_j \right]
\]

\[
\geq \liminf_n \left[ \|P_k x\|_j^2 + \|Q_{k+1} x_n\|_j^2 \right]^{1/2} - \|Q_k x\|_j
\]

\[
= \left[ \|P_k x\|_j^2 + \liminf_n \|Q_{k+1} x_n\|_j^2 \right]^{1/2} - \|Q_k x\|_j
\]

\[
= \left[ \|P_k x - Q_{k+1} x_n\|_j^2 + \liminf_n \|x_n\|_j^2 \right]^{1/2} - \|Q_k x\|_j
\]

for \( k = 1, 2, \ldots \). Hence we obtain the following inequality

\[
\liminf_n \|x - x_n\|_j \geq \lim_k \left\{ \|P_k x\|_j^2 + \liminf_n \|x_n\|_j^2 \right\}^{1/2} - \|Q_k x\|_j
\]

\[(\ast)\]

which leads to (3). In other words, \((J, \| \cdot \|_j)\) has the uniform \(w^*\)-Opial property for \( j = 1, 2 \).

\[\square\]

**Corollary 1.** For \( j = 1, 2 \) the space \((J, \| \cdot \|_j)\) has the uniform Opial property.

**Remark 1.** The uniform \(w^*\)-Opial property of \((J, \| \cdot \|_j)\), \( j = 1, 2 \), implies the following important property of these spaces. The \((w^*)\)-modulus of noncompact convexity of a (dual) Banach space \((X, \| \cdot \|)\) is the function \(\Delta_x : [0, 1] \to [0, 1]\) \((\Delta^*_x : [0, 1] \to [0, 1])\) defined by

\[
\Delta_x(\varepsilon) = \inf\{1 - \operatorname{dist}(0, A)\}
\]

\[
(\Delta^*_x(\varepsilon) = \inf\{1 - \operatorname{dist}(0, A)\}),
\]

where the infimum is taken over all convex (weak* compact convex) subsets \( A \) of the closed unit ball with \( \chi(A) \geq \varepsilon \), and \( \chi \) is the Hausdorff measure of noncompactness [4]. In the case of \((J, \| \cdot \|_j)\), \( j = 1, 2 \), the inequality \((\ast)\) implies \(\Delta^*_x(\varepsilon) > 0\).
for all $\varepsilon > 0$. This means that these spaces are $\Delta^*$-uniformly convex and every weakly* compact convex subset $C$ of $(J, \| \cdot \|_j)$ $(j = 1, 2)$ has a compact asymptotic center [4]. Taking $A = \text{conv} \{ u_n \}$, where $u_n = \sum_{i=1}^{n}$, we see that
\[ \chi(A) = 1 \]
and
\[ \forall x \in A \| x_j \| = 1 \]
($j = 1, 2$). Therefore $\Delta_x \equiv 0$ for $X = (J, \| \cdot \|_j)$, $j = 1, 2$.

Here we have to mention that generally the uniform Opial property does not imply the $\Delta$-uniform convexity as the following example shows.

**Example ([23]).** For $\lambda > 1$ let $X$ be the space $l_2$ with the norm
\[ \| (\alpha_n) \| = \max \{ \lambda |\alpha_1|, \| (\alpha_n) \|_2 \} \]
where $\| \cdot \|_2$ is the norm in $l_2$. Then
\[
\liminf_n \| x_n - x \| = \max \left\{ \lambda |\alpha_1|, \left( \liminf_n \| x_n \|_2^2 + \| x \|_2^2 \right)^{1/2} \right\}
\geq (1 + \| x \|_2^2)^{1/2} \geq (1 + \lambda^{-2} \| x \|_2^2)^{1/2}
\]
for each $x \in X$ and each sequence $\{x_n\}$ with $w\text{-}\lim_n x_n = 0$ and $\liminf \| x_n \| \geq 1$.

This inequality guarantees the uniform Opial property of $X$, but $\Delta_x(\varepsilon) = 0$ for all $\varepsilon \leq (1 - \lambda^{-2})^{1/2}$.

**Remark 2.** It is easy to observe that James Tree $JT$ constructed by R.C. James [11] also has the $w^*$-uniform Opial property, where $JT$ is the dual space to the Banach space $B$ generated by the biorthogonal functionals $\{f_{n,i}\}$ to the basis $\{e_{n,i}\}$ (this basis is analogous to the basis $\{u_n\}$ in $J$) given in [19]. The proof of this fact is a slight modification of the proof of the Theorem. Corollary 1 and Remark 1 are also valid for $JT$. (See also [13] for the $w^*$-Opial property.)

**Remark 3.** The uniform ($w^*$-) Opial property of $(J, \| \cdot \|_j)$ with $j = 1, 2$ and $JT$ implies that these spaces satisfy the weak (weak*) uniform Kadec-Klee property [16].

We conclude our paper with three corollaries.

**Corollary 2.** $(J, \| \cdot \|_j)$, $j = 1, 2$, and $JT$ have weak and weak* normal structure.

**Corollary 3.** $(J, \| \cdot \|_j)$, $j = 1, 2$, and $JT$ have the FPP for weakly* compact convex subsets.

Recall that a Banach space $(X, \| \cdot \|)$ is said to satisfy the ($w^*$-) demiclosedness principle [1], [4] if whenever $C$ is a nonempty weakly (weakly*) compact convex subset of $X$ and $T : C \to X$ is nonexpansive, then the mapping $I - T$, where $I$ is the identity operator, is ($w^*$-) demiclosed, i.e. if $\{x_n\}$ is weakly (weakly*) convergent to $x$ and $\{x_n - Tx_n\}$ converges strongly to $y$, then $x - Tx = y$. It is known that every Banach space with the ($w^*$-) Opial property satisfies the ($w^*$-) demiclosedness principle.
Corollary 4. $(J, \| \cdot \|_j)$, $j = 1, 2$, and $JT$ satisfy the $(w^*)$ demiclosedness principle.

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