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The area formula for $W^{1,n}$-mappings

**Jan Malý**

**Abstract.** Let $f$ be a mapping in the Sobolev space $W^{1,n}(\Omega, \mathbb{R}^n)$. Then the change of variables, or area formula holds for $f$ provided removing from counting into the multiplicity function the set where $f$ is not approximately Hölder continuous. This exceptional set has Hausdorff dimension zero.

**Keywords:** Sobolev spaces, change of variables, area formula, Hölder continuity

**Classification:** 28A75, 26B15

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $f: \Omega \rightarrow \mathbb{R}^n$ be a mapping and $S \subset \Omega$. We define the multiplicity function (Banach indicatrix) $\mathcal{N}$ by

$$\mathcal{N}(y, f, S) = \sharp \{x \in S : f(x) = y\}.$$

If $f$ is a Lipschitz mapping on $\Omega$, then $f$ is differentiable a.e. (Rademacher theorem) and the area formula

$$\int_S |\det \nabla f(x)| \, dx = \int_{\mathbb{R}^n} \mathcal{N}(y, f, S) \, dy$$

holds for any measurable set $S \subset \Omega$ (see [2]). The same is true if $f$ is a continuous representative of a mapping in $W^{1,p}(\Omega, \mathbb{R}^n)$ with $p > n$ (as the Lusin (N)-property holds, cf. Proposition 1.1 and [1]). There are continuous mappings in $W^{1,p}(\Omega, \mathbb{R}^n)$ with $p \leq n$ for which the area formula does not hold (see [13], [9] and references therein). The problem of the area formula for Sobolev mappings is continuously stimulating. For interesting recent results we refer to [10]. One approach consists in looking for “partial area formulae”: a set $S_0$ of full measure is found such that

$$\int_S |\det \nabla f(x)| \, dx = \int_{\mathbb{R}^n} \mathcal{N}(y, f, S \cap S_0) \, dy$$

for all measurable $S \subset \Omega$.

As shown by Federer [3], for any $f$ which has partial derivatives almost everywhere there are sequences $f_j$ of Lipschitz mappings and $M_j$ of disjoint measurable sets such that $f_j = f$ on $M_j$ and $\Omega \setminus \bigcup_j M_j$ has zero measure. The following proposition is then an easy and well known consequence (cf. [10], [5]).

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1.1 Proposition. Let $S \subset \Omega$ be a measurable set and $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$. Then the following assertions are equivalent:

(i) “the (N)-property holds for $f$ on $S$”: $|f(E)| = 0$ for each $E \subset S$ with $|E| = 0$,

(ii) “the area formula holds for $f$ on $S$”:
$$\int_{S'} |\det \nabla f(x)| \, dx = \int_{\mathbb{R}^n} N(y, f, S') \, dy$$
for each measurable set $S' \subset S$,

(iii) “the change of variables formula holds for $f$ on $S$”:
$$\int_{S} u(f(x)) |\det \nabla f(x)| \, dx = \int_{\mathbb{R}^n} u(y) N(y, f, S) \, dy$$
for each nonnegative Borel measurable function $u$ on $\mathbb{R}^n$.

Let $f$ be a function in $L^1_{\text{loc}}$ defined a.e. in $\Omega$. Then the function $\tilde{f}(x) := \lim_{r \to 0^+} \int_{B(x, r)} f$.

is defined in $\Omega$ except a set of measure zero. The function $\tilde{f}$ is called the Lebesgue representative of $f$ and we say that $f$ is Lebesgue precise if $f = \tilde{f}$. If, in addition, $f \in W^{1,p}(\Omega)$ with $p > 1$, then $\tilde{f}$ is defined up to a set of p-capacity zero and p-finely continuous except for a set of p-capacity zero (see [14, Section 3.3]), which means that it is p-quasi-continuous ([6, Theorem 8]). These references are also recommended for definitions of p-capacity, p-quasi-continuity and p-fine topology (by the p-capacity $\text{cap}_p$ we understand the Bessel capacity denoted by $B_{1,p}$ in [14]).

The following theorem gives a good choice of a set of canonical nature for which the area formula holds ([5], cf. also [4]).

1.2 Proposition. Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ be Lebesgue precise and $S$ be the set of all points of $\Omega$ at which $f$ is approximately differentiable. Then the area formula holds for $f$ on $S$.

The aim of this paper is to use a slight refinement of methods from [9] to show that for $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ the set to be removed for validity of the area formula can be found even smaller. The following theorem will be proved in the next section.

1.3 Theorem. Suppose that $f$ is an $n$-quasi-continuous representative of a mapping in $W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ and $S$ is the set of all points of $\Omega$ at which $f$ is approximately Hölder continuous. Then the area formula holds for $f$ on $S$.

The size of the exceptional set is estimated in the following result, proved in Section 3.
1.4 Theorem. Suppose that $f$ is an $n$-quasi-continuous representative of a mapping in $W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ and $S$ is the set of all points of $\Omega$ at which $f$ is approximately Hölder continuous. Then the set $\Omega \setminus S$ has Hausdorff dimension zero.

2. Points of approximate Hölder continuity

Let $f$ be a measurable function on $\Omega$. We say that $f$ is \textit{approximately Hölder continuous} at $x \in \Omega$ if there is $\alpha \in (0, 1]$ and a set $M$ such that

$$\limsup_{y \to x, y \in M} \frac{|f(y) - f(x)|}{|y - x|^{\alpha}} < \infty$$

and the Lebesgue density of $M$ at $x$ is one.

We need the following version of the Gehring oscillation lemma.

2.1 Lemma. Let $f$ be a quasi-continuous representative of a mapping in $W^{1,n}(B(x,r), \mathbb{R}^m)$. Then for almost all $t \in (0, r)$ the restriction of $f$ to $\partial B(x,t)$ is a continuous representative of an element of $W^{1,n}(\partial B(x,t), \mathbb{R}^m)$ and the inequality

$$(\text{diam } f(\partial B(x,t)))^n \leq c t \int_{\partial B(x,t)} |\nabla f|^n dS.$$  

holds.

Proof: The estimate follows from the Sobolev inequality and a similarity argument if $f$ is $C^1$. In the general case there are $C^1$ mappings $f_j$ such that

$$\sum_j \|f_j - f\|_{1,p}^p < \infty.$$  

Using integration over radii it follows that there is $N_1 \subset (0, r)$ with $|N_1| = 0$ such that

$$\sum_j \int_{\partial B(x,t)} (|f_j - f|^p + |\nabla f_j - \nabla f|^p) dS < \infty.$$  

Since $f$ is $n$-quasi-continuous and $f_j \to f$ in $W^{1,n}$, we know (after selecting a subsequence) that $f_j \to f$ except a set $E$ of $n$-capacity zero ([12, Theorem 5.4]). By well known relations between capacity and Hausdorff measure ([12]) it follows that the linear measure of $E$ is zero, so that there is $N_2 \subset (0, r)$ with $|N_2| = 0$ such that $f_j \to f$ everywhere on $\partial B(x,t)$ for each $t \in (0, r) \setminus N_2$. If $t \in (0, r) \setminus (N_1 \cup N_2)$, then the Sobolev inequality implies uniform convergence $f_j \to f$ on $\partial B(x,t)$ and a routine passage to limit yields (2.1).

The following tool is essentially Lemma 4.5 of [9].
2.2 Lemma. Let $B$ be a ball $B(x,r)$ and $\tau \in (0, 1)$. Suppose $f \in W^{1,n}(B, \mathbb{R}^n)$. Then there is a measurable set $A \subset B$ such that

$$|B \setminus A| \leq \tau |B|$$

and

$$(\text{diam } f(A))^n \leq c \int_A (1 + |\nabla f|^n) \, dy,$$

c depends only on $n$ and $\tau$.

Proof: Although the proof in fact follows the idea in [9], we present it here, as in the original the mapping $f$ is assumed to be continuous. Set

$$\omega_0 = \inf \{\text{diam } f(\partial B(x,t)) : t \in [r/2, r]\}.$$

Find $t_0 \in [r/2, r]$ such that $\text{diam } f(\partial B(x,t_0)) < \omega_0 + r$ and choose $z_0 \in \partial B(x,t_0)$. Denote

$$u(y) = |f(y) - f(z_0)|,$$

$$\lambda_i = i(r + \omega_0),$$

$$E_i = \{y \in B(x,r) : |f(y) - f(z_0)| < \lambda_i\},$$

$$v_{i,j} = \max(\min(u, \lambda_j), \lambda_i).$$

First we will derive the estimate

$$\omega_0^n \leq c \int_{B(x,r) \cap E_3} |\nabla f|^n \, dy. \tag{2.2}$$

If there exists $t_1 \in [r/2, r]$ such that $\partial B(x,t_1) \cap E_2 = \emptyset$, we observe that $v_{1,2} = \lambda_1$ on $\partial B(x,t_0)$ and $v_{1,2} = \lambda_2$ on $\partial B(x,t_1)$. It follows

$$(r + \omega_0)^n = (\lambda_2 - \lambda_1)^n \leq c \int_{B(x,r)} |\nabla v_{1,2}|^n \, dy \leq c \int_{E_3 \cap B(x,r)} |\nabla f|^n \, dy,$$

which proves (2.2) in this case. Now we may assume that $\partial B(x,t)$ intersects $E_2$ for all $t \in [r/2, r]$. We write $F = \{t \in [r/2, r] : \partial B(x,t) \subset E_3\}$. Using Lemma 2.1 to $f$ and $v_{2,3}$, for almost all $t \in [r/2, r]$ we get

$$t^{-1} \omega_0^n \leq c \int_{E_3 \cap \partial B(x,t)} |\nabla f|^n \, dS. \tag{2.3}$$

Indeed, if $t \in F$, we estimate

$$\omega_0^n \leq (\text{diam } f(\partial B(x,t)))^n \leq c t \int_{\partial B(x,t)} |\nabla f|^n \, dS$$

$$= c t \int_{E_3 \cap \partial B(x,t)} |\nabla f|^n \, dS,$$
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while for $t \in [r/2, r] \setminus F$ we have

$$\omega^n_0 \leq (\lambda_3 - \lambda_2)^n \leq (\text{diam } v_{2,3}(\partial B(x,t)))^n \leq c t \int_{\partial B(x,t)} |\nabla v_{2,3}|^n dS \leq c t \int_{E_3 \cap \partial B(x,t)} |\nabla f|^n dS.$$ 

Integrating (2.3) over $t \in [r/2, r]$ we obtain (2.2). Now, if $|B \setminus E_3| \leq \tau |B|$, we are done. Indeed, setting $A = B \cap E_3$, we obtain

$$(\text{diam } f(A))^n \leq c(\omega_0 + r)^n \leq c \int_A |\nabla f|^n dy + c r^n \leq c \int_A (1 + |\nabla f|^n) dy.$$ 

Otherwise we find $k \geq 3$ such that

$$|B \setminus E_{k+1}| \leq \tau |B| < |B \setminus E_k|$$ 

and set $A = B \cap E_{k+1}$. Since $v_{1,k} - \lambda_1 = 0$ on $\partial B(x,t_0)$, a Poincaré-type inequality ([14, Section 4.5]) yields

$$(\text{diam } f(A))^n \leq (2\lambda_{k+1})^n \leq c \tau^{-1} r^{-n} \int_B (v_{1,k} - \lambda_1)^n dy \leq c \tau^{-1} \int_B |\nabla v_{1,k}|^n dy \leq c \tau^{-1} \int_A |\nabla f|^n dy,$$

which concludes the proof.

\[\square\]

**Proof of Theorem 1.3:** We verify the condition (i) of Proposition 1.1. Recall that $S$ is the set of all points where $f$ is approximately H"older continuous. Let $E \subset S$ be a set of zero measure. Decomposing $E$ if necessary into a countable union, we may assume that $f$ is $1/m$-H"older continuous at all $x \in E$ for $m \in \mathbb{N}$ fixed. Choose an open set $G \subset \Omega$ containing $E$. Fix $x \in E$. There are $K > 0$ and a set $M \subset \Omega$, such that the Lebesgue density of $M$ at $x$ is one and

$$|f(y) - f(x)| \leq K|y - x|^{1/m}$$ 

for all $y \in M$. Find $r_0 > 0$ such that $B(x,r_0) \subset G$ and

$$|B(x,r) \setminus M| \leq \frac{1}{4}|B(x,r)|$$

for all $r \in (0,r_0)$. For $k = 0, 1, \ldots$ we denote

$$r_k = r_0 2^{-k}, \\
B_k = B(x, r_k)$$
and using Lemma 2.2 we find a measurable set \( A_k \subset B_k \) such that

\[
|B_k \setminus A_k| \leq 2^{-n-2}|B_k|
\]

and

\[
(2.4) \quad (\text{diam } f(A_k))^n \leq c \int_{A_k} (1 + |\nabla f|^n) \, dy.
\]

Then (for \( k \geq 1 \))

\[
|B_k \setminus A_{k-1}| \leq 2^{-n-2}|B_{k-1}| = \frac{1}{4}|B_k|,
\]

\[
|B_k \setminus A_k| \leq 2^{-n-2}|B_k|,
\]

\[
|B_k \setminus M| \leq \frac{1}{4}|B_k|,
\]

and thus there is \( x_k \in A_k \cap A_{k-1} \cap M, x_k \neq x \). We have

\[
(2.5) \quad |f(x_k) - f(x)| \leq c|x_k - x|^{1/m} \leq c2^{-k/m}.
\]

Choosing \( b \) with \( (1 + 1/b)^{-1} > 2^{-1/m} \) we claim that the set

\[
I(x) = \{ k \in \mathbb{N} : |f(x_{k+1}) - f(x)| \leq b|f(x_k) - f(x_{k+1})| \}
\]

is infinite. Indeed, assuming that \( \max I(x) = k_0 \), we get

\[
|f(x_k) - f(x)| \leq |f(x_{k+1}) - f(x)| + |f(x_k) - f(x_{k+1})| \leq (1 + 1/b)|f(x_{k+1}) - f(x)|
\]

for each \( k > k_0 \), which leads to a contradiction with (2.5), as an iteration yields

\[
|f(x_k) - f(x)| \geq c(1 + 1/b)^{-k}.
\]

Denote

\[
R_k(x) = \text{diam } f(A_k) + |f(x_{k+1}) - f(x)|.
\]

Since \( x_k, x_{k+1} \in A_k \), we have

\[
f(A_k) \subset B(f(x), R_k)
\]

and (using (2.4))

\[
R_k(x) \leq \text{diam } f(A_k) + b|f(x_{k+1}) - f(x_k)| \leq (1 + b) \text{ diam } f(A_k)
\]

\[
\leq c(1 + b) \left( \int_{A_k} (1 + |\nabla f|^n) \, dy \right)^{1/n}
\]
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defined whenever $k \in I(x)$. For the next step of the proof we write this estimate in the form

$$\begin{align*}
(R_k(x))^n & \leq c \int_{f^{-1}(B(f(x), R_k(x)))} (1 + |\nabla f|^n) \, dy, \quad k \in I(x).
\end{align*}$$

The balls $B(f(x), R_k)$, $x \in E$, $k \in I(x)$, form a Vitali cover of $f(E)$. By the Vitali covering theorem there is a disjoint subcover $B(f(x_\tau), R_\tau)$ such that the set $N := E \setminus \bigcup \tau B(f(x_\tau), R_\tau)$ has zero Lebesgue measure. Using (2.6) it follows that

$$|f(E)| \leq |N| + 2^n \sum_\tau R_\tau^n \leq c \sum_\tau \int_{f^{-1}(B(f(x_\tau), R_\tau^n))} (1 + |\nabla f|^n) \, dy \leq c \int_G (1 + |\nabla f|^n) \, dy.$$ 

Varying the set $G$ we get the required conclusion $|f(E)| = 0$. \hfill \Box

3. Size of the exceptional set

3.1 Lemma. Let $p > 1$. Then the $p$-fine closure of any open set $A \subset \mathbb{R}^n$ has the same $p$-capacity as $A$.

Proof: See [11, Proposition 3.2]. \hfill \Box

The following tool is an immediate consequence of Theorem 7 of [8].

3.2 Proposition. Let $f$ be an $n$-quasi-continuous representative of a mapping in $W^{1,n}(\Omega, \mathbb{R}^n)$, $\varepsilon > 0$ and $p < n$. Then there is an open set $G \subset \mathbb{R}^n$ such that $\text{cap}_p G < \varepsilon$ and the restriction of $f$ to $\Omega \setminus G$ is locally Hölder continuous.

Proof of Theorem 1.4: Using Proposition 3.2 we find open sets $G_{i,j}$ and $p_i \nearrow n$ such that $\text{cap}_{p_i}(G_{i,j}) < 1/j$ and $f$ is locally Hölder continuous on $\Omega \setminus G_{i,j}$. Denote by $S_{i,j}$ the $p_i$-fine interior of $\mathbb{R}^n \setminus G_{i,j}$ and $S = \Omega \setminus \bigcup_{i,j} S_{i,j}$. By Lemma 3.1, $\text{cap}_{p_i}(\mathbb{R}^n \setminus S_{i,j}) < 1/j$, and thus the Hausdorff dimension of $E$ is zero (see e.g. [7, Theorem 2.26]). Consider a point $x \in S$. Then there are $i, j$ such that $x \in \Omega \cap S_{i,j}$. Since the set $S_{i,j}$ is $p_i$-finely open and $f$ is locally Hölder continuous on $S_{i,j}$, it follows that the Lebesgue density of $S$ at $x$ is one ([14, Section 3.3]) and $f$ is approximately Hölder continuous at $x$. \hfill \Box

References


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