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On isometric embeddings of Hilbert’s cube into c

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Abstract. In our note, we prove the result that the Hilbert’s cube equipped with l_p –metrics, $p \geq 1$, cannot be isometrically embedded into c .

Keywords: Lipschitz embeddings, Hilbert’s cube

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1. Introduction

Aharoni [1] proved that every separable metric space can be Lipschitz embedded into c_0 . His proof was simplified by Assouad [2] who also improved the Lipschitz constants given by Aharoni’s construction. This fact was further generalized by Pelant in [3] using the theorem that metric spaces uniformly homeomorphic to subspaces of some $c_0(\kappa)$ are exactly those satisfying the A.H. Stone paracompactness theorem in a uniform way, i.e. in which for any uniform cover, one can find a uniform refinement which is locally finite. Some further improvements of Lipschitz constants were given in [3]. For Lipschitz embeddings of compact metric spaces into c_0 , these improvements give the best possible estimates, i.e. for any compact metric space (X, d) and any $\varepsilon > 0$, there is $F : X \rightarrow c_0$, s.t.

$$\frac{1}{1 + \varepsilon} d(x, y) \leq \|F(x) - F(y)\|_{c_0} \leq d(x, y) \text{ for each } x, y \in X.$$

On the other hand, it is shown in [3] that the Hilbert’s cube equipped with l_1 –metrics cannot be isometrically embedded into c_0 .

In our note, we prove the analogous result for the Hilbert’s cube endowed by l_p –metrics, $p \geq 1$ and the space c . Moreover, we show that there exists a compact subset of c which cannot be isometrically embedded into c_0 , i.e. there is a non-formal difference between c and c_0 .

2. Notation and results

Let I be a closed unit interval $[0, 1]$ and as usually $I^{\mathbb{N}_0}$ be the Hilbert’s cube. For $p \geq 1$, $I^{\mathbb{N}_0}$ constitutes the metric space $I_p = (I^{\mathbb{N}_0}, \rho_p)$ by the metric ρ_p , where for each $x, y \in I^{\mathbb{N}_0}$

$$\rho_p(x, y) = \left(\sum_{i=1}^{\infty} \frac{|x_i - y_i|^p}{2^i} \right)^{\frac{1}{p}}.$$

Let c be the set of all real sequences $x = \{\xi_n\}$ such that finite $\lim_{n \rightarrow \infty} \xi_n = \xi_\infty$ exists, endowed by the norm $\|x\| = \sup_n |\xi_n|$. On a normed linear space $(c, \|\cdot\|)$ we consider the induced metric $\sigma(x, y) = \|x - y\|$. A subspace c_0 consists of all sequences $x = \{\xi_n\}$ such that $\lim_{n \rightarrow \infty} \xi_n = 0$. In the metric space X , $B_X(r, s)$ denotes the closed ball of the center $r \in X$ and the radius s , $S_X(r, s)$ denotes its sphere. By $x = \{\xi\}$ we mean a constant sequence. Recall that using Ascoli-Arzelà Theorem, we have a characterization of a relatively compact infinite subset of c .

Proposition. *A set $\{x_\alpha\}_{\alpha \in A} = \{\{\xi_{\alpha,m}\}\}_{\alpha \in A}$ is a relatively compact subset of c , if the following two conditions are satisfied:*

$$\{x_\alpha\}_{\alpha \in A} \text{ is equi-bounded, i.e., } \sup_{\alpha \in A} \|x_\alpha\| < \infty,$$

$$\{x_\alpha\}_{\alpha \in A} \text{ is uniformly convergent, i.e., } \lim_{n \rightarrow \infty} \sup_{\substack{\alpha \in A \\ m \geq n}} |\xi_{\alpha,m} - \xi_{\alpha,\infty}| = 0.$$

□

Theorem 1. *There exists a compact set K in c which cannot be isometrically embedded into c_0 .*

PROOF: Let K contain $\{0\}$ and the sequences $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ of elements of c defined by the equalities

- (i) $a_k = \{\alpha_l\}$, $\alpha_k = 1 + \frac{1}{2^k}$, $\alpha_l = 1$ for $l \neq k$,
- (ii) $b_k = \{\beta_l\}$, $\beta_l = -\alpha_l$ for each l .

By Proposition, the reader can easily verify that K is a compact subset of c and for different positive integers k, l , we have from (i), (ii)

$$\text{(iii) } \sigma(a_k, a_l) = \sigma(b_k, b_l) = \frac{1}{2^{\min(k,l)}}, \quad \sigma(a_k, b_l) = 2 + \frac{1}{2^{\min(k,l)}}$$

$$\sigma(a_k, \{0\}) = \sigma(b_k, \{0\}) = 1 + \frac{1}{2^k}, \quad \sigma(a_k, b_k) = 2 + \frac{1}{2^{k-1}}.$$

Suppose that an isometry F from K into c_0 exists. Without loss of generality we can assume that $\{0\}$ is a fixed point of an isometry F . Denote all images in $F(K)$ by ‘tilde’, i.e. $F(K) = \tilde{K}$ and $F(a_k) = \tilde{a}_k$ for $a_k \in K$. Since the property of F , \tilde{K} is a compact subset of $c_0 \subset c$ and an analogous equalities as (iii) can be written for elements of \tilde{K} . By Proposition \tilde{K} is uniformly convergent and there exists a positive integer k_0 such that for each $\tilde{x} = \{\tilde{\xi}_n\} \in \tilde{K}$

$$\text{(iv) } \sup_{n > k_0} |\tilde{\xi}_n| < \frac{1}{2}.$$

Consider a pair $\tilde{a}_k = \{\tilde{\alpha}_l\}$, $\tilde{b}_k = \{\tilde{\beta}_l\}$ from \tilde{K} . Since $\sigma(\tilde{a}_k, \tilde{b}_k) = 2 + \frac{1}{2^{k-1}}$ there exists $l_1 \in \{1, 2, \dots, k_0\}$ such that

$$\text{(v) } |\tilde{\alpha}_{l_1} - \tilde{\beta}_{l_1}| = 2 + \frac{1}{2^{k-1}}, \quad |\tilde{\alpha}_{l_1}| = |\tilde{\beta}_{l_1}| = 1 + \frac{1}{2^k}.$$

Because \tilde{K} is infinite and the condition (iv) holds, we have the equalities (v) with an index l_1 for infinitely many $\{k_i\}$ and pairs $\tilde{a}_{k_i}, \tilde{b}_{k_i}$. Then for $i \neq j$ either $\sigma(\tilde{a}_{k_i}, \tilde{a}_{k_j}) = 2 + \frac{1}{2^{k_i}} + \frac{1}{2^{k_j}}$ or $\sigma(\tilde{a}_{k_i}, \tilde{b}_{k_j}) = 2 + \frac{1}{2^{k_i}} + \frac{1}{2^{k_j}}$. Since F is distance-preserving and by (iii) we have a contradiction. □

Theorem 2. *There is no isometric embedding of $I_p = (I^{\aleph_0}, \rho_p)$ to (c, σ) .*

PROOF: To the contrary suppose that such an isometric embedding $F : I_p \rightarrow c$ exists. Without loss of generality we can assume that $F(\{\frac{1}{2}\}) = \{0\}$. Using a notation stated above we can write

$$(vi) \quad B_{I_p}(\{\frac{1}{2}\}, \frac{1}{2}) = I_p, F(S_{I_p}(\{\frac{1}{2}\}, \frac{1}{2})) \subset S_c(\{0\}, \frac{1}{2}).$$

It is clear from the definitions of metrics ρ_p, σ that

$$(vii) \quad S_{I_p} = S_{I_p}(\{\frac{1}{2}\}, \frac{1}{2}) = \{0, 1\}^{\aleph_0}, S_c = S_c(\{0\}, \frac{1}{2}) \subset [-\frac{1}{2}, \frac{1}{2}]^{\aleph_0},$$

$$(viii) \quad \text{for any } x \in S_{I_p} \text{ there exists a single opposite } y \in S_{I_p} \text{ with } \rho_p(x, y) = 1.$$

In what follows we shall denote this opposite element by x' . The sphere S_c can be divided to three disjoint sets K, L, M by the way $K = \{x \in S_c, \lim |x_i| < \frac{1}{2}\}$, $L = \{x \in S_c, \lim x_i = \frac{1}{2}\}$, $M = \{x \in S_c, \lim x_i = -\frac{1}{2}\}$. Note that

$$(ix) \quad \text{card}\{i, |x_i| = \frac{1}{2}\} < \infty \text{ for each } x \in K.$$

Let for $x \in S_c$, $E_+(x) = \{i, x_i = \frac{1}{2}\}$, $E_-(x) = \{i, x_i = -\frac{1}{2}\}$ and define on S_c the equivalence relation by the following : elements $x, y \in S_c$ are equivalent if and only if $E_+(x) = E_+(y)$ and $E_-(x) = E_-(y)$. According to (ix) there is a countable set of the equivalence classes τ_k which forms the decomposition $\{\tau_k\}$ of K . So, we have

$$(x) \quad S_c = (\cup \tau_k) \cup L \cup M.$$

Because of (vii) the set S_{I_p} is uncountable, hence we have from (vi), (x) that one of the following cases must be realized :

I. There is a positive integer k such that $\text{card}(\tau_k \cap F(S_{I_p})) \geq 2$.

Choose different $a, b \in (\tau_k \cap F(S_{I_p}))$. Since σ is a metric and a, b are equivalent ($\in K$) the relations

$$(xi) \quad 0 < \sigma(a, b) < 1$$

hold. By (viii) for $x \in S_{I_p}$, $x = F^{-1}(a)$, there exists $x' \in S_{I_p}$ with the property $\rho_p(x, x') = 1$. If we put $d = F(x')$, then

$$\sigma(a, d) = \rho_p(x, x') = 1,$$

hence by (xi) $b \neq d$. Now, the reader can easily verify that because of $E_+(a) = E_+(b)$ and $E_-(a) = E_-(b)$ we even have

$$\sigma(b, d) = \rho_p(F^{-1}(b), x') = 1.$$

This implies $F^{-1}(b) = F^{-1}(a) = x$, hence we have $a = b$. But this is a contradiction.

II. The set $L \cap F(S_{I_p})$ is uncountable and $\text{card}(\tau_k \cap F(S_{I_p})) \leq 1$ for every positive integer k .

Then first of all by (viii), $M \cap F(S_{I_p}) = \emptyset$ and for all but countably many $a \in (L \cap F(S_{I_p}))$ an opposite element $a'' = F((F^{-1}(a))')$ which is guaranteed by (viii) belongs again to $L \cap F(S_{I_p})$. Thus, if we define for $n \in \mathbb{N}$ the sets G_n by

$$G_n = \left\{ a \in (L \cap F(S_{I_p})), \inf_{i \geq n+1} a_i > -\frac{1}{2} \ \& \ \inf_{i \geq n+1} a''_i > -\frac{1}{2} \right\},$$

there exists $m \in \mathbb{N}$ for which G_m is infinite (uncountable). Similarly as above the reader can easily see that there exist two different elements a, b in G_m such that $E_+(a) = E_+(b)$ and $E_-(a) = E_-(b)$, i.e.

$$\sigma(a, a'') = \sigma(b, a'') = 1.$$

Hence we have a contradiction.

III. The set $M \cap F(S_{I_p})$ is uncountable and $\text{card}(\tau_k \cap F(S_{I_p})) \leq 1$ for every positive integer k .

This case is analogous to the previous one.

The proof of Theorem 2 is finished. □

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